

Relative sea level change in Western Alaska as constructed from satellite altimetry and repeat GPS measurements

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Thesis Defense

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Western Alaska LCC



Outline

- **Definition of relative sea level (RSL)**
- **Motivation**
- **Oceanic and tectonic setting in Western Alaska**
- **Data**
 - Tide gauge
 - Historic water levels
 - Satellite altimetry
 - GPS
- **Results**
 - Satellite altimetry
 - GPS vertical velocity model
 - Glacial isostatic adjustment (GIA) model
 - Closing the circle using tide gauges
 - RSL model
- **Summary**
- **Future Work**

What is RELATIVE sea level change?

- RSL = *level of the sea* minus *level of the land*
- Both of these terms can vary regionally
- In Alaska, land level changes are, in general, more rapid than sea level changes.

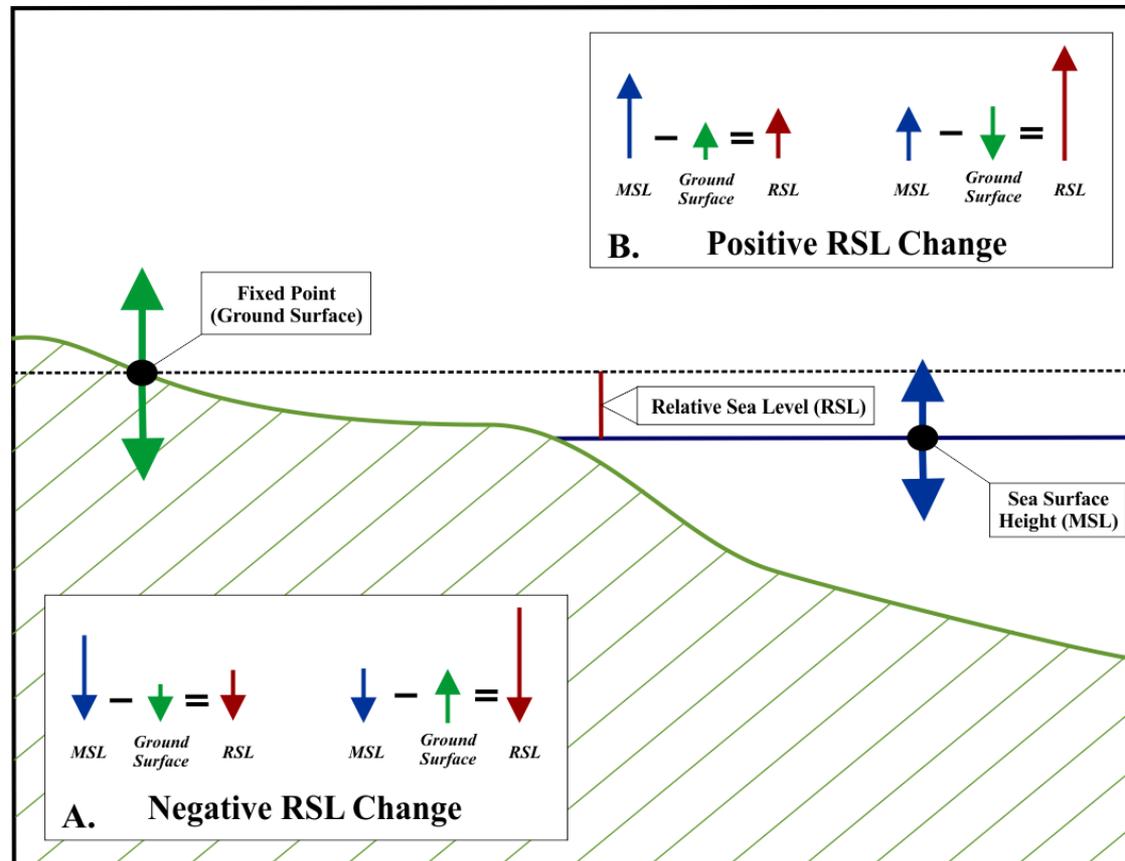
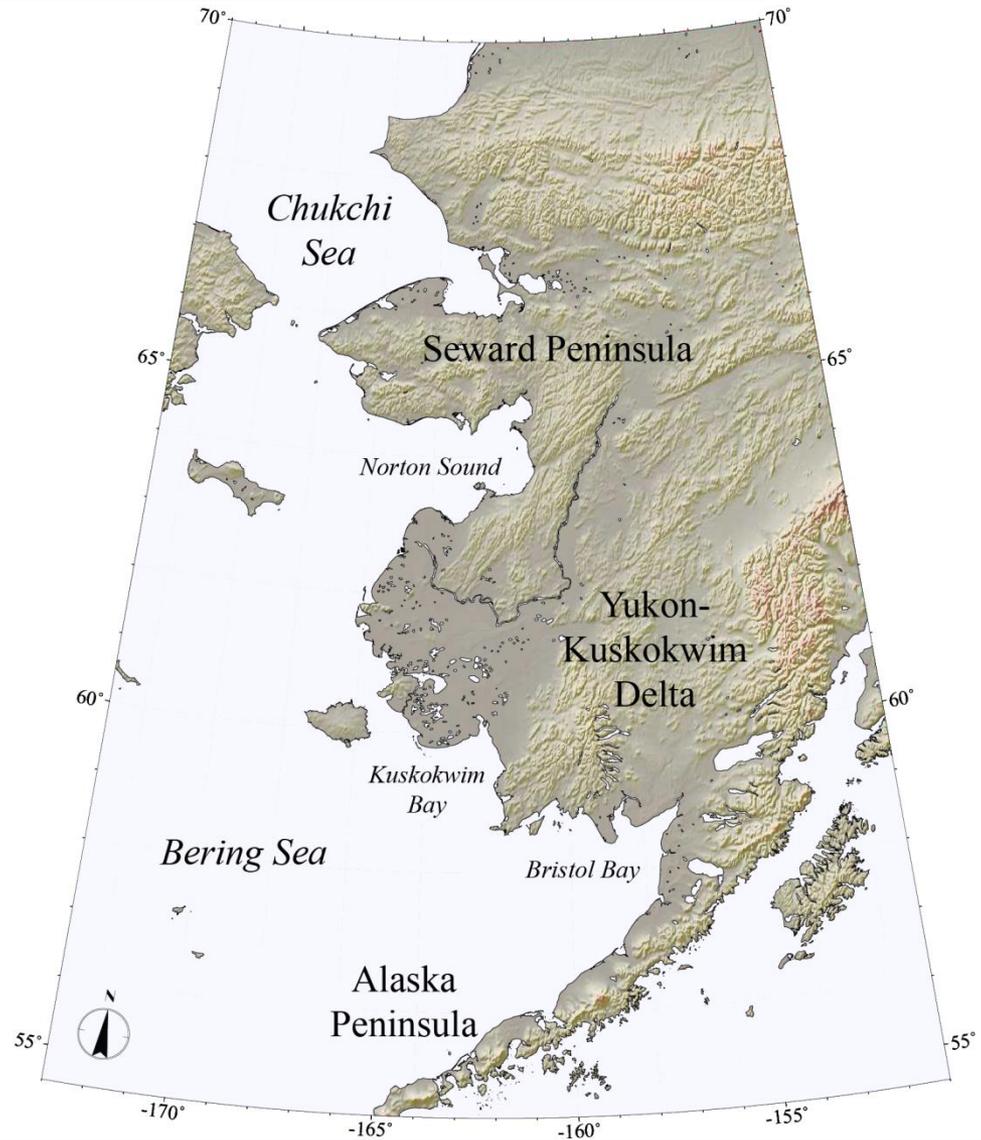


Figure 1. Diagram of RSL and different combinations of the vertical velocities of the onshore and offshore components

Western Alaska

- Coastal communities with populations from 100 to 4,000 people.
- Communities are situated in vulnerable coastal environments, such as sand spits and barrier islands.
- Population is predominantly Native Alaskans of various tribes.
- No road system, travel is restricted to airplane or boat.

Figure 2. Study area referred to as “Western Alaska”



Motivation

- Erosion, storm surge, and flooding are causing severe destruction to communities on almost yearly time scales.
- The state and individual communities face decisions that involve responses that range from local erosion control to relocation of entire villages.
- To make these decisions they need to be well informed of the trend and projected effects that relative sea level will have on their communities.

Figure 3. Erosion of coastal bluff in Meshik, AK in September 2013. Meshik has been abandoned and residents relocated to Port Heiden, AK.



Motivation



Figure 4. Flooding in Golovin, AK November 2012.

Photo courtesy of John Peterson

Motivation

Other Benefits

- RSL change can inundate flat, low lying, saturated lands with salt water that changes vegetation cover, salmon spawning, and fresh water and land resources for larger mammals.
- RSL change reflects other changes in climate, atmosphere, geophysics, and oceanography, so it is beneficial to start a baseline for monitoring it.

Figure 5. ASTER image of the Yukon Delta and sea ice in the southern half of Norton Sound

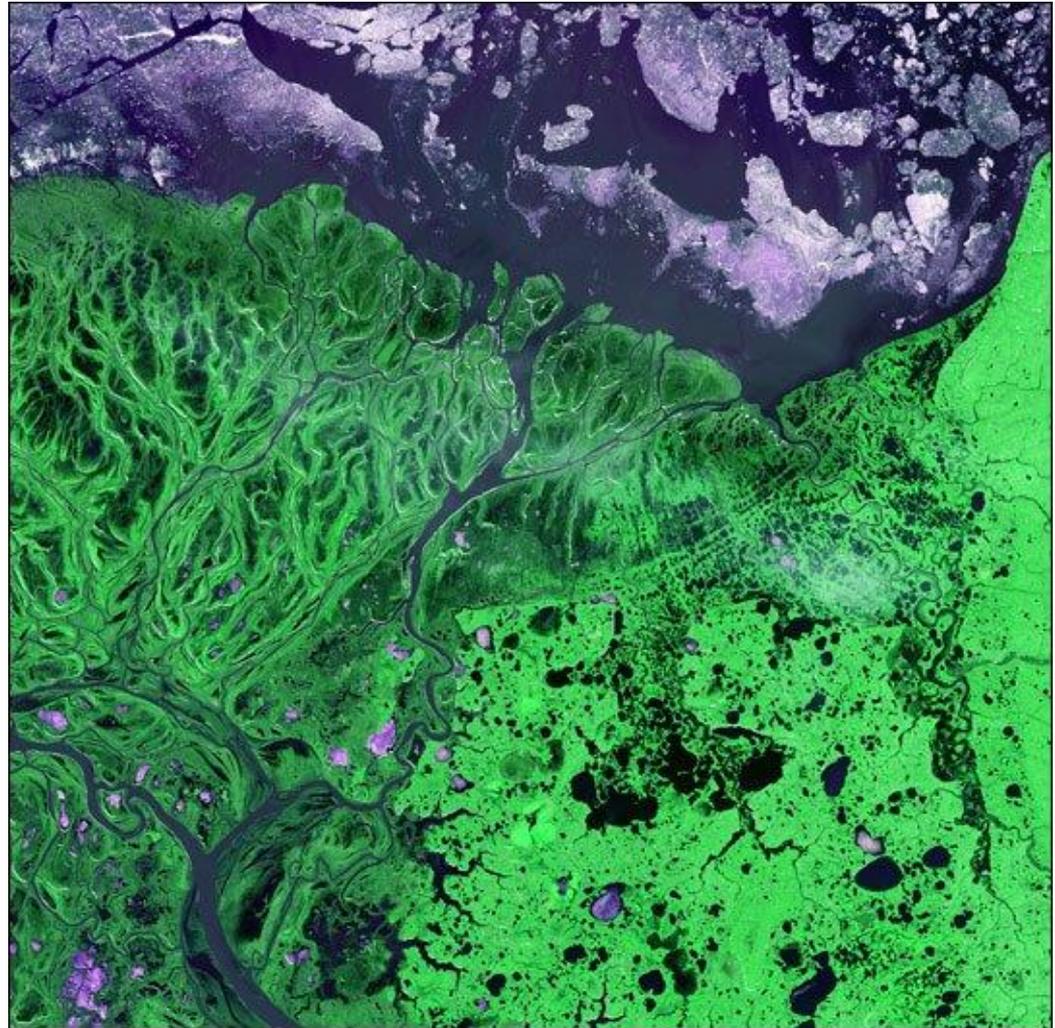


Image provided by the USGS EROS Data Center Satellite Systems Branch as part of the Earth as Art II image series.

Oceanic Processes

Thermal expansion - Mass balance - Salinity and circulation changes

- Vulnerable coastal environments: sand spits, barrier islands, deltas, and tidal flats.
- Offshore shelf with shallow bays and sounds.
- Complex basin geometry can have big effects.

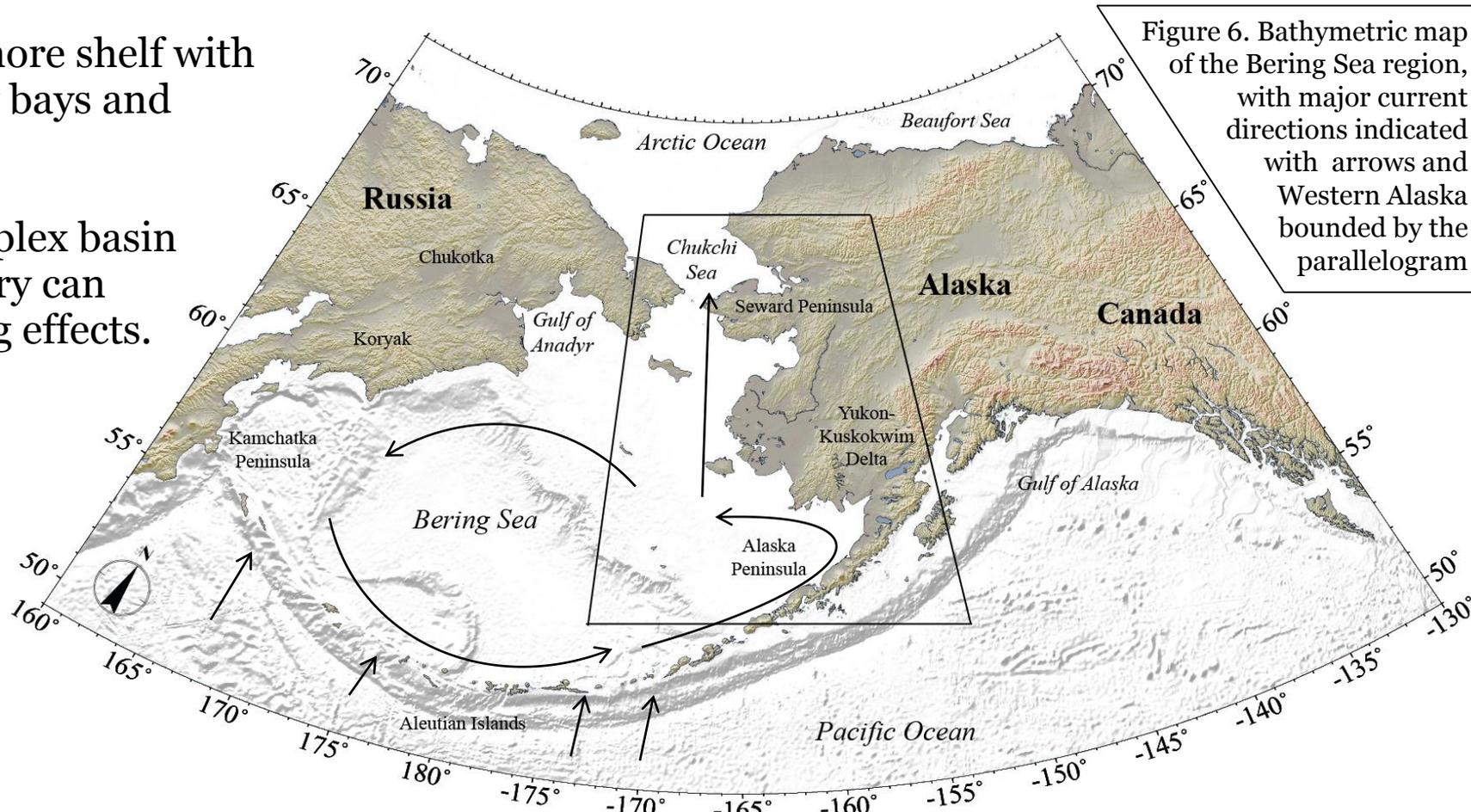
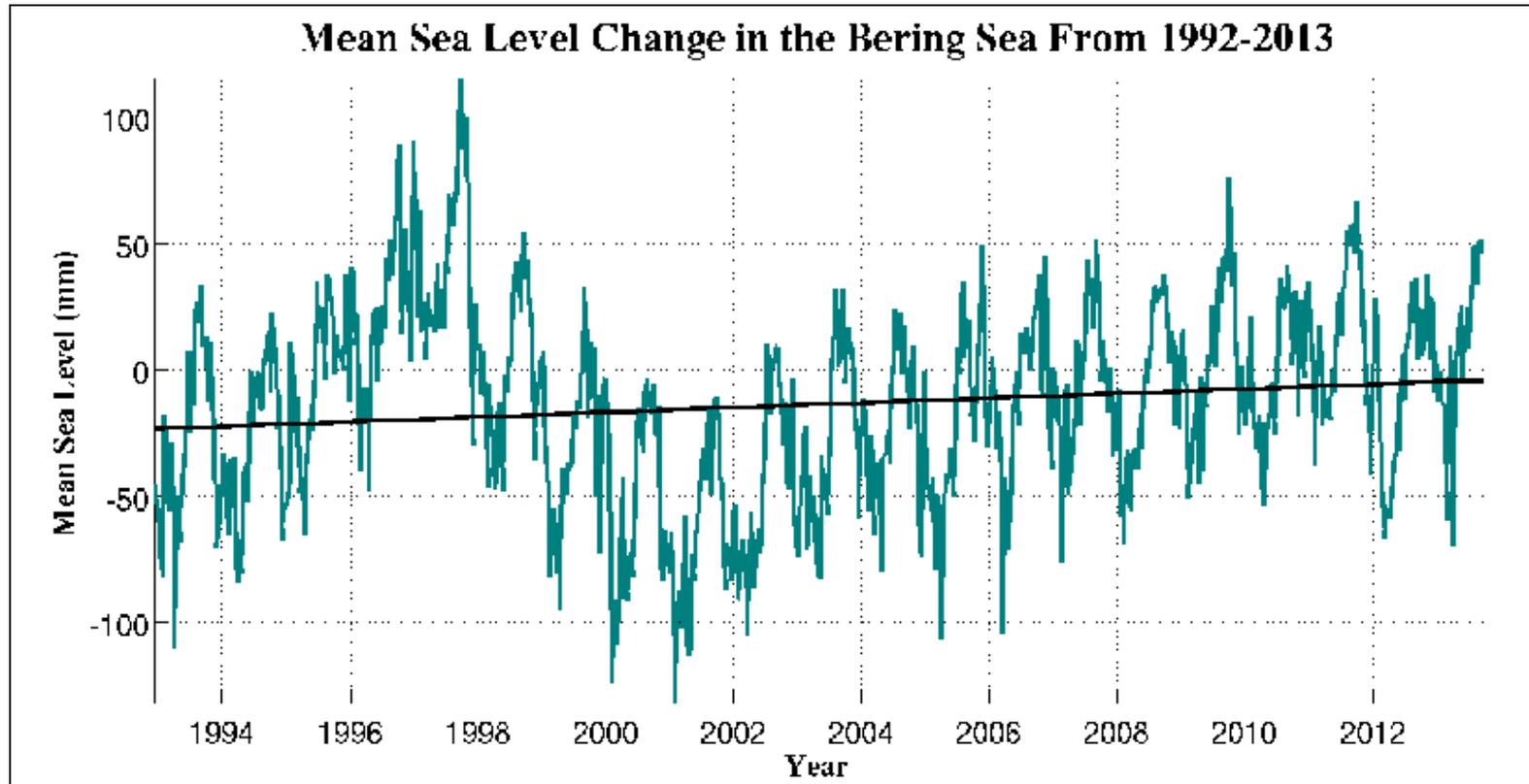


Figure 6. Bathymetric map of the Bering Sea region, with major current directions indicated with arrows and Western Alaska bounded by the parallelogram

Sea Level Trends

Global Average = **+3.2 +/- 0.4 mm/yr** (Nerem, 2010; Leuliette and Willis, 2011)
Bering Sea Regional Average = **+2.6 mm/yr** (Nerem et al., 2013)

Figure 7. Bering sea, satellite altimetry non-gridded data set shows 2.6 mm/yr of sea level rise over 20 years (Nerem et al., 2013) .



Tectonic Setting

- Proposed boundary of the Bering, Eurasian, Pacific, and North American plates (Cross and Freymueller, 2008).
- Horizontal movement is clockwise rotation.
- No convergence or divergence in Western Alaska.

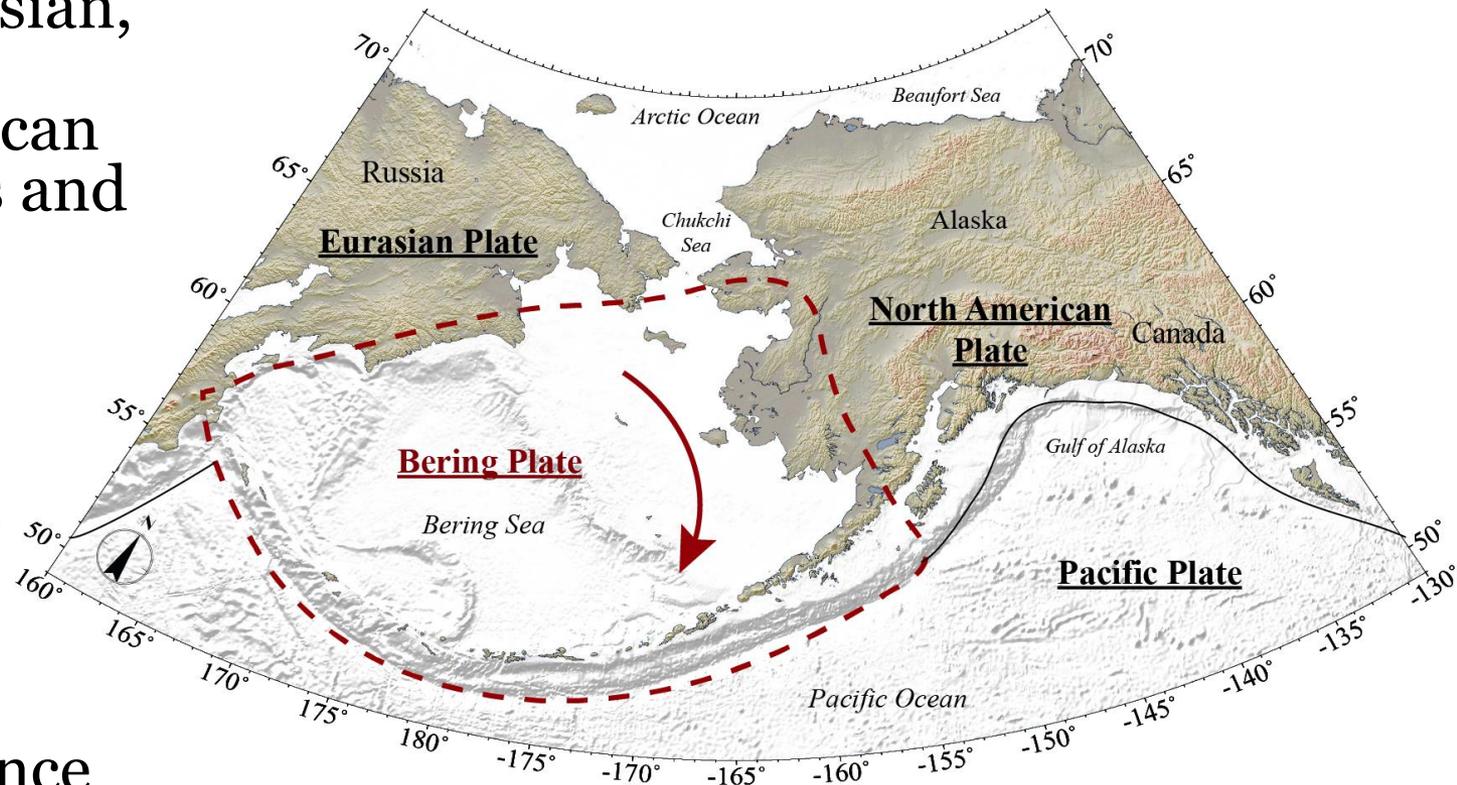


Figure 8. Bering plate (Cross and Freymueller, 2008) is dashed red outline and rotation is indicated with arrow.

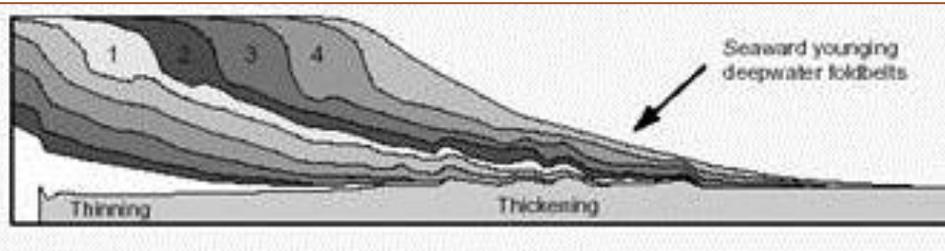
Tectonic processes

Glacial isostatic adjustment (GIA)

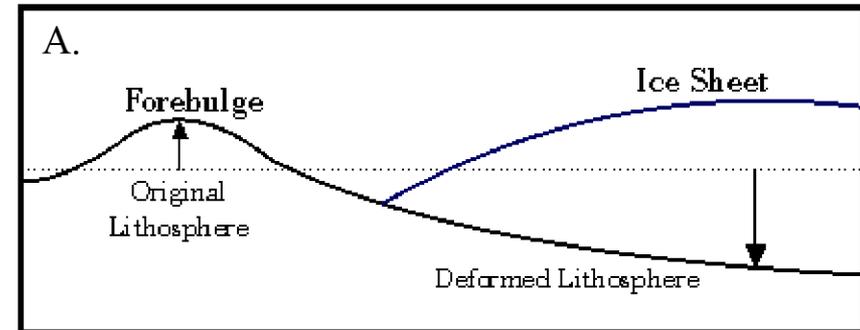
Figure 9. Example of GIA (right), uplift directly under the ice sheet, but subsidence of the forebulge.

Delta Subsidence

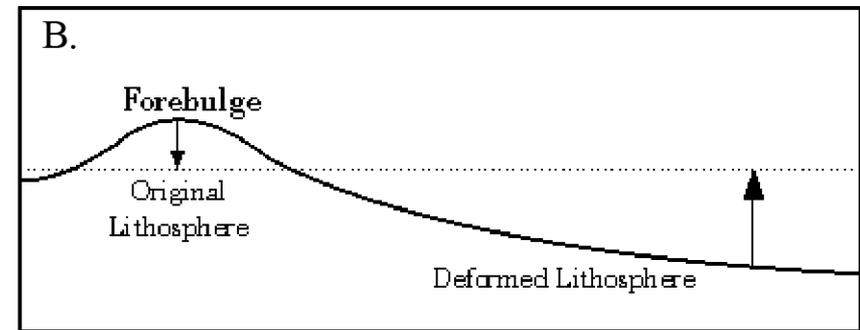
Figure 10. Prograding delta (below) illustrating sediment deposition and loading. (Albertz et al., 2006).



During Glaciation



After Glaciation



Tectonic processes - GIA

- 2 main GIA estimations in Alaska define forebulge as coastal or inland.
- Earth models run for Southern Alaska, but not Northern or Western Alaska
- Relevant deformation results from Laurentide and Cordilleran ice sheets

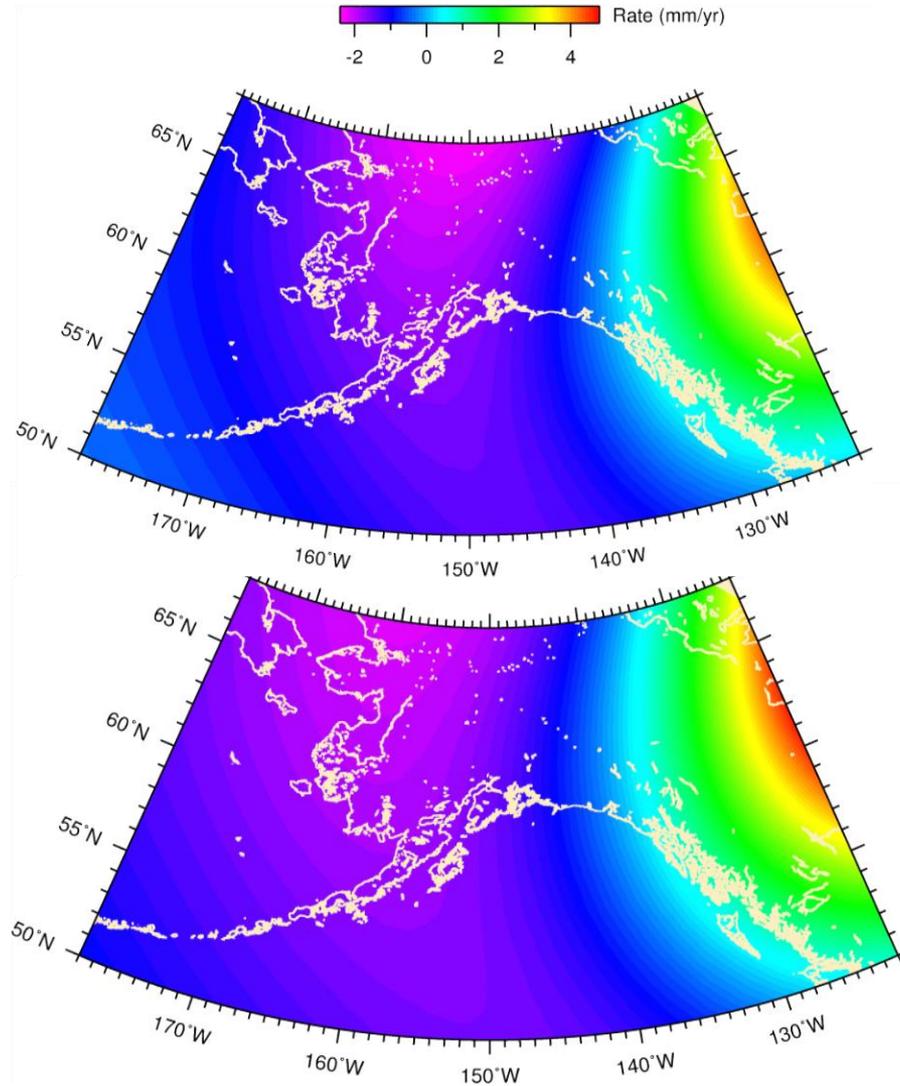


Figure 11. Tectonic vertical velocity based on ICE-3G and Continental shield-type earth model.

Figure 12. Tectonic vertical velocity based on ICE-3G and Alaska-type earth model.

Tectonic processes – Delta subsidence

Figure 13. Map of the Yukon River drainage basin

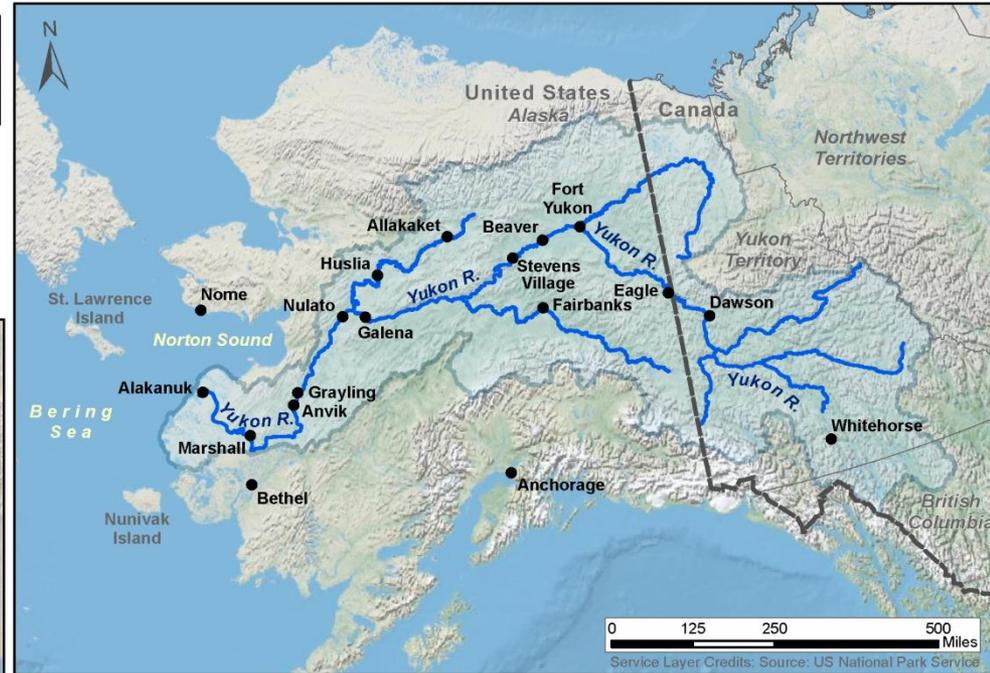
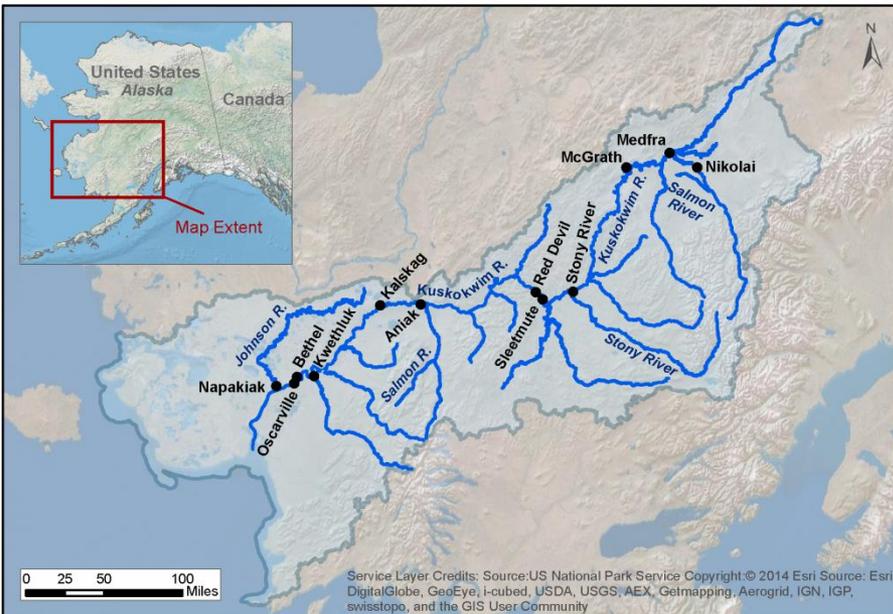


Figure 14. Map of the Kuskokwim River drainage basin



Tectonic processes – Delta Subsidence



Figure courtesy of John Wallace

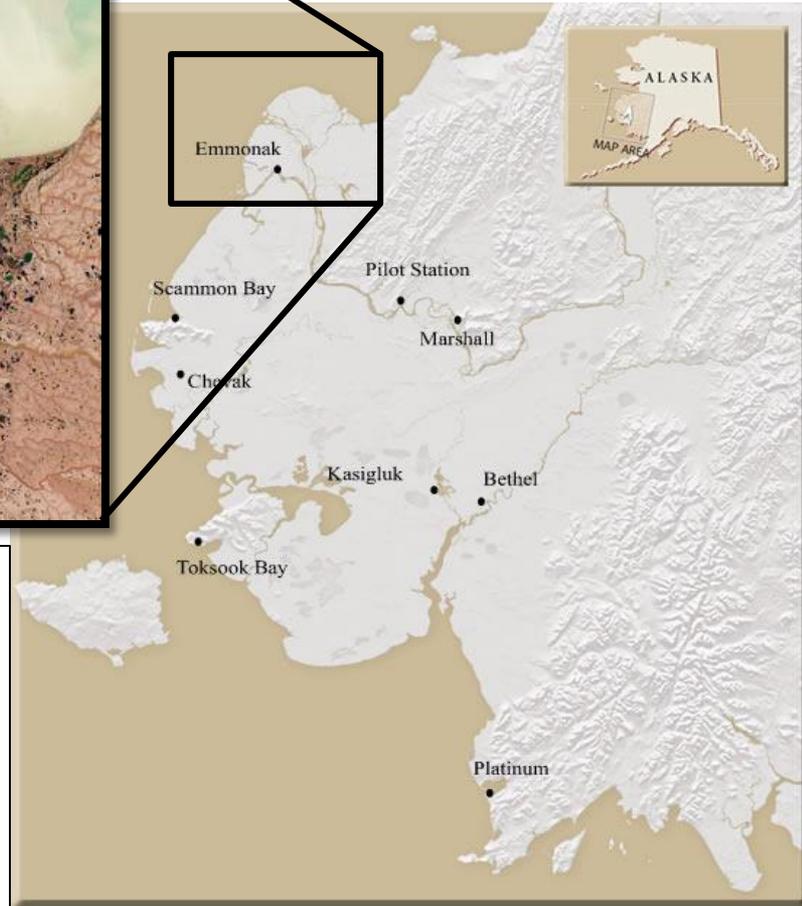


Figure 15. Map of the Yukon-Kuskokwim delta (right). Natural-color image (above) of the Yukon River delta from NASA Earth Observatory, image created by Jesse Allen and Robert Simmon, using Landsat data provided by the United States Geological Survey.

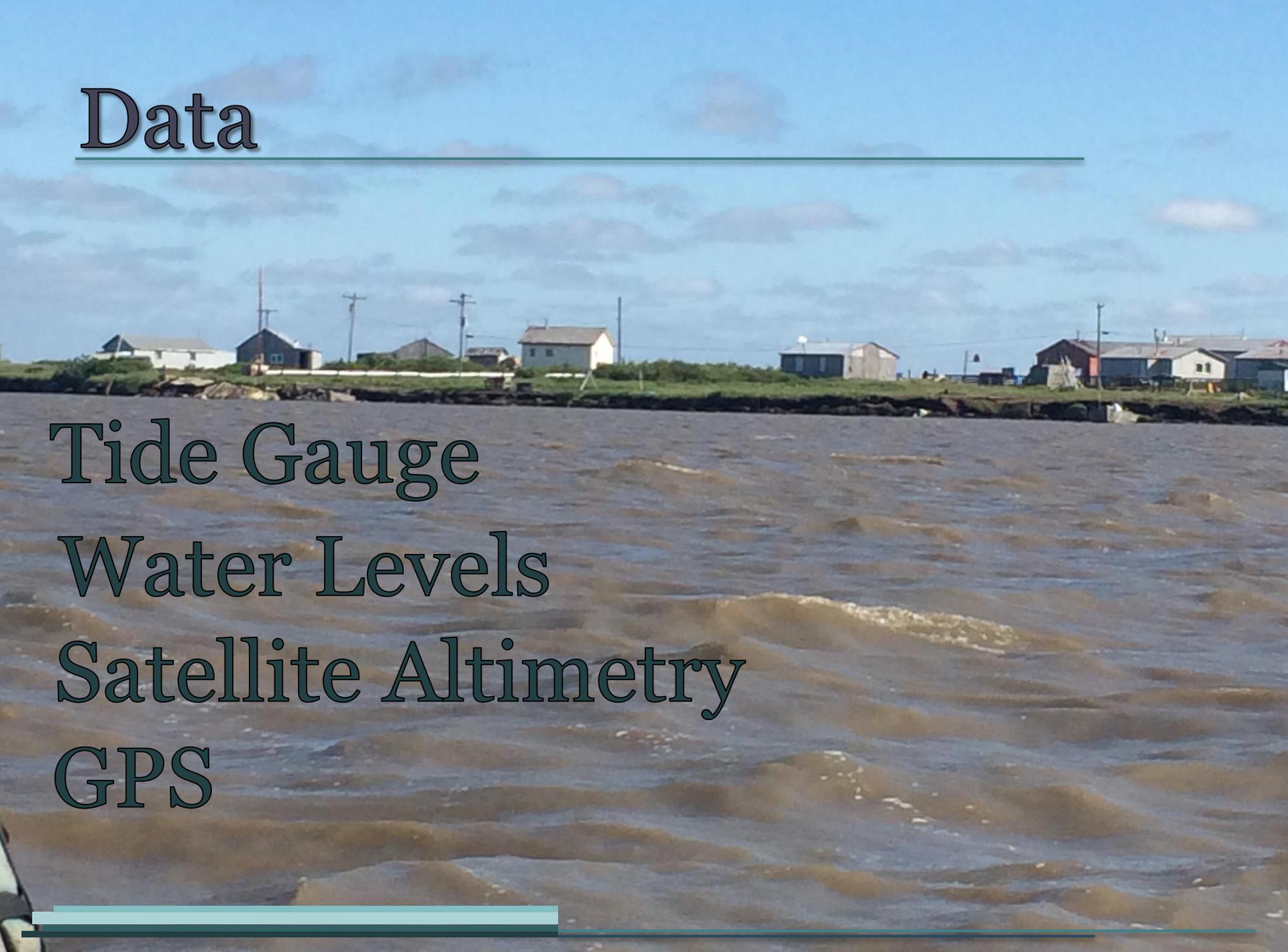
Data

Tide Gauge

Water Levels

Satellite Altimetry

GPS



Data – Tide gauge

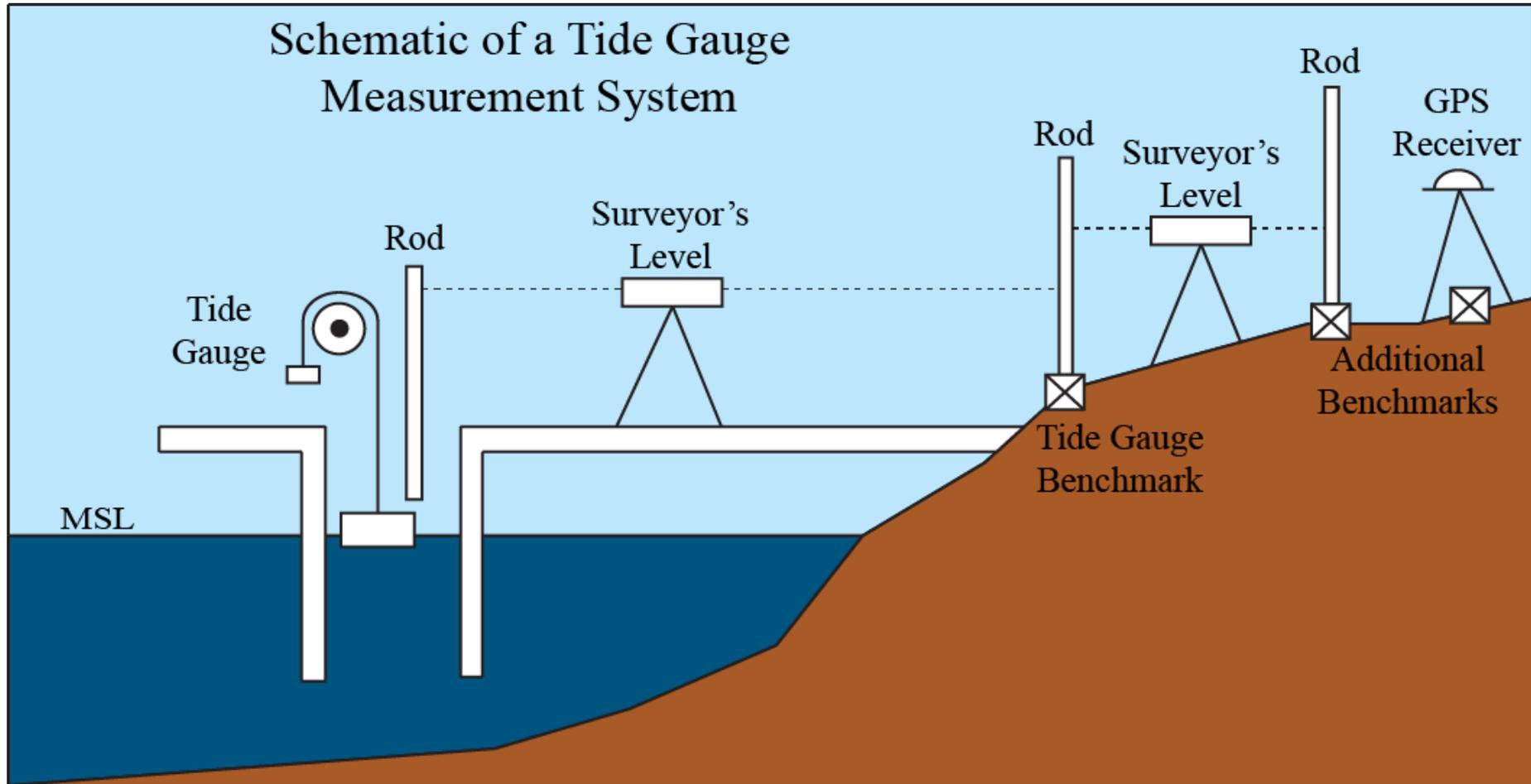
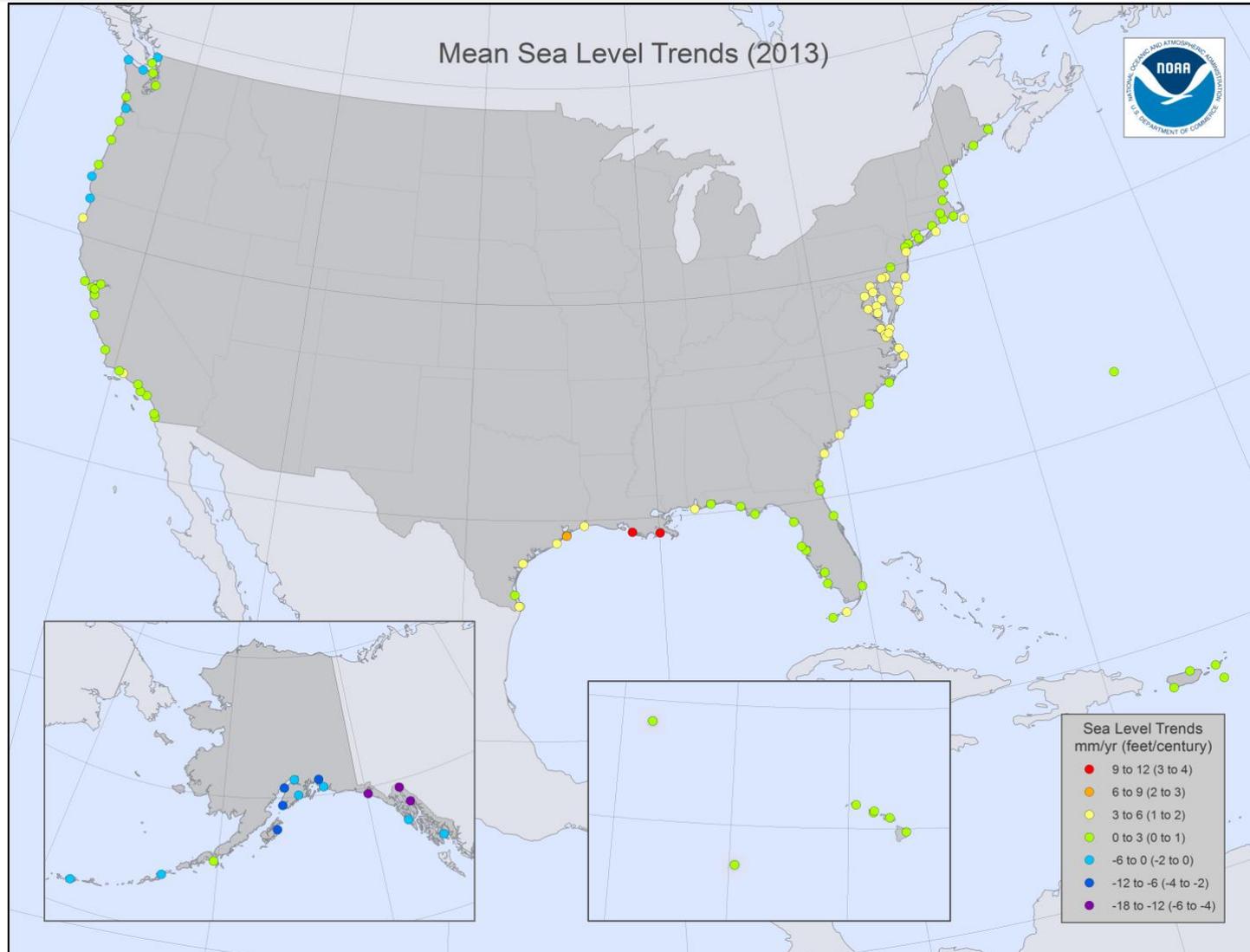


Figure 16. Diagram of how a tide gauge measures water levels and ties them to a fixed land surface reference point to measure RSL.

Data – Tide gauge

- US has good coastal coverage of tide gauges and therefore many published RSL rates
- Western and Northern Alaska have no published rates



Data – Tide gauge

- Data from each tide gauge available at PSMSL and NOAA, relative to a variety of water level datums.
- To compare these data, a comparison between each of these datasets was done for tide gauges in Nome, Seldovia and Sand point.

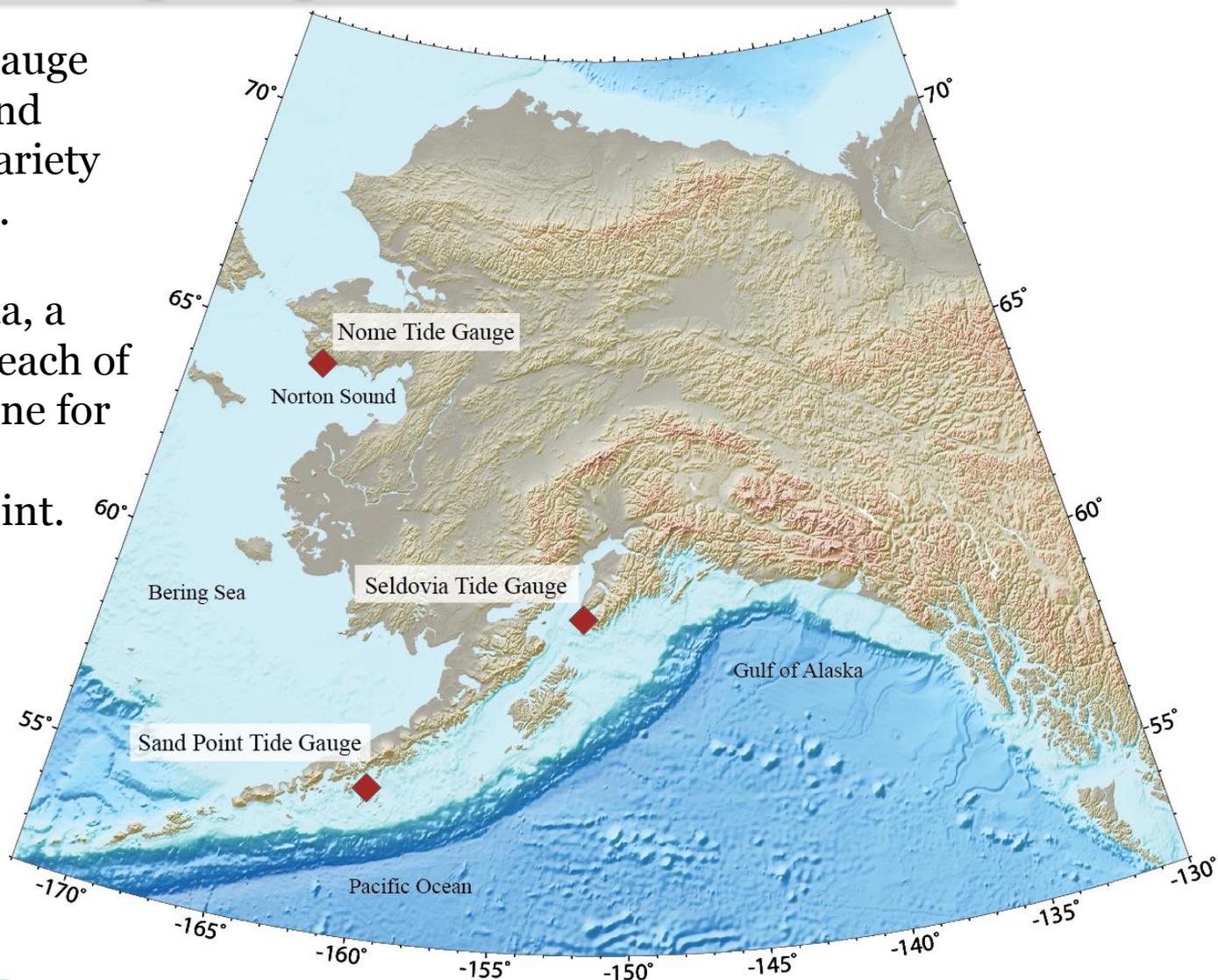


Figure 18. Locations of published RSL trends provided by NOAA. Water level measurements are from NOAA tide gauges located at each dot .

Data – Tide gauge

Location of Data	Type of data	Trend (mm/yr)		
		Nome	Sand Point	Seldovia
PSMSL	MSL - Monthly	0.91	0.84	-9.65
NOAA	MSL - Monthly	0.5	0.56	-10.41
NOAA	MLLW - Monthly	1.32	0.44	-10.79
NOAA	MSL Trend - Monthly	NA	0.68	-10.36
NOAA	MLLW - Hourly 1	0.02	NA	NA
Calculated in this study	MLLW - Monthly 1	0.37	NA	NA
NOAA	MLLW - Hourly 2	0.65	NA	NA
Calculated in this study	MLLW - Monthly 2	0.52	NA	NA
NOAA	MSL - Published	-0.48*	0.38	-10.47
NOAA	MSL - Published Error	+/- 4.24*	+/- 0.97	+/- 0.85

Figure 19. Locations of published RSL trends provided by NOAA. Water level measurements are from NOAA tide gauges located at each dot. Asterisk indicates that values are not publically published, but were obtained through personal communication

Data – Tide gauge

- Continuous station in Nome as of 1992, but with large gap until 1997.
- Rate calculated in this study to be **+0.50 mm/yr** using linear least squares fit to the data.
- No published value from NOAA is available, but personal communication with Dr. Zervas at NOAA provided their estimate of **-0.48 +/- 4.24 mm/yr**

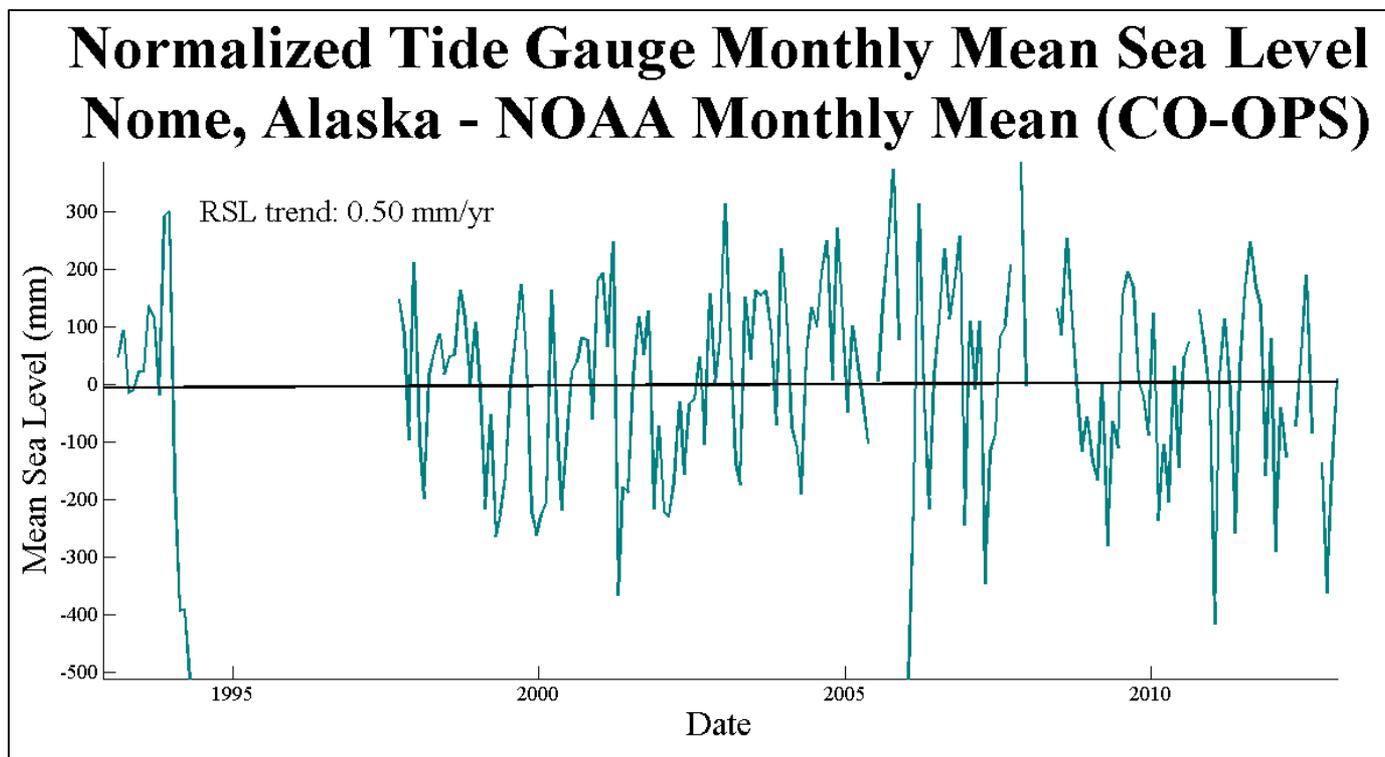


Figure 20.
Monthly mean
tidal data
from the
Nome tide
gauge (NOAA,
2014).

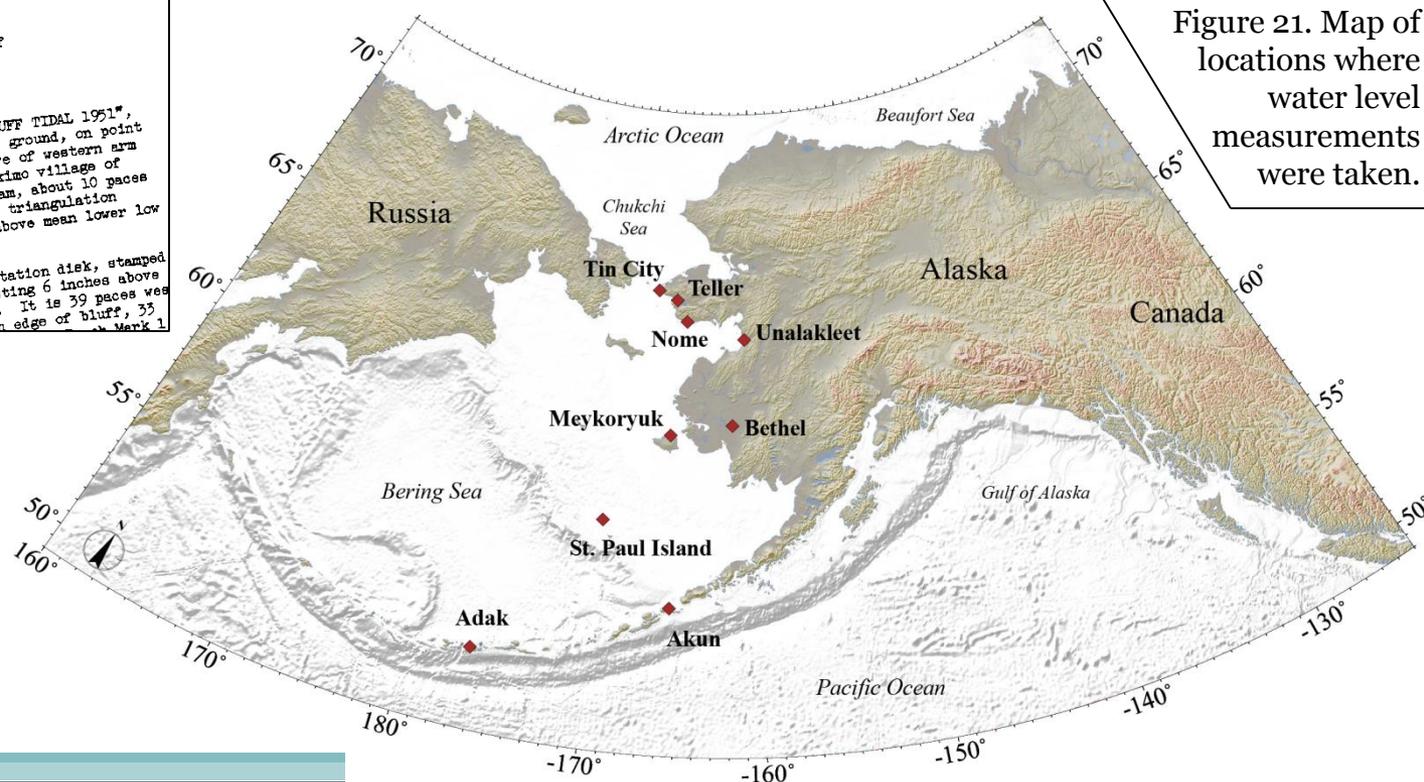
Data – Water levels

- Water levels measured with staff: 1950s – present
- Tied to benchmarks that have been lost or destroyed
- Very wide range of measurements with unrealistic and inconsistent rates, **between -265 mm/yr and +24.1 mm/yr**

13/52
46-6298
WESTERN AND NORTHERN ALASKA - 33
U. S. COAST AND GEODETIC SURVEY
TIDAL BENCH MARKS
Nelson Island (Bluff, 6 miles southeast of
Tumukak), Toksook Bay
Lat. 60° 31' ; Long. 165° 08'

BENCH MARK 1 (1951) is a standard disk, stamped "BLUFF TIDAL 1951", set in top of 2-inch soil pipe projecting 8 inches above ground, on point of land on north shore of Toksook Bay, on southern shore of western arm of Nelson Island and about 3 miles east-northeast of eskimo village of Unikumute. It is about 300 feet southwest of small stream, about 10 paces west of edge of 30-foot bluff and 300 feet northwest of triangulation station BLUFF (Bench Mark 2). Elevation: 38.16 feet above mean lower low water.

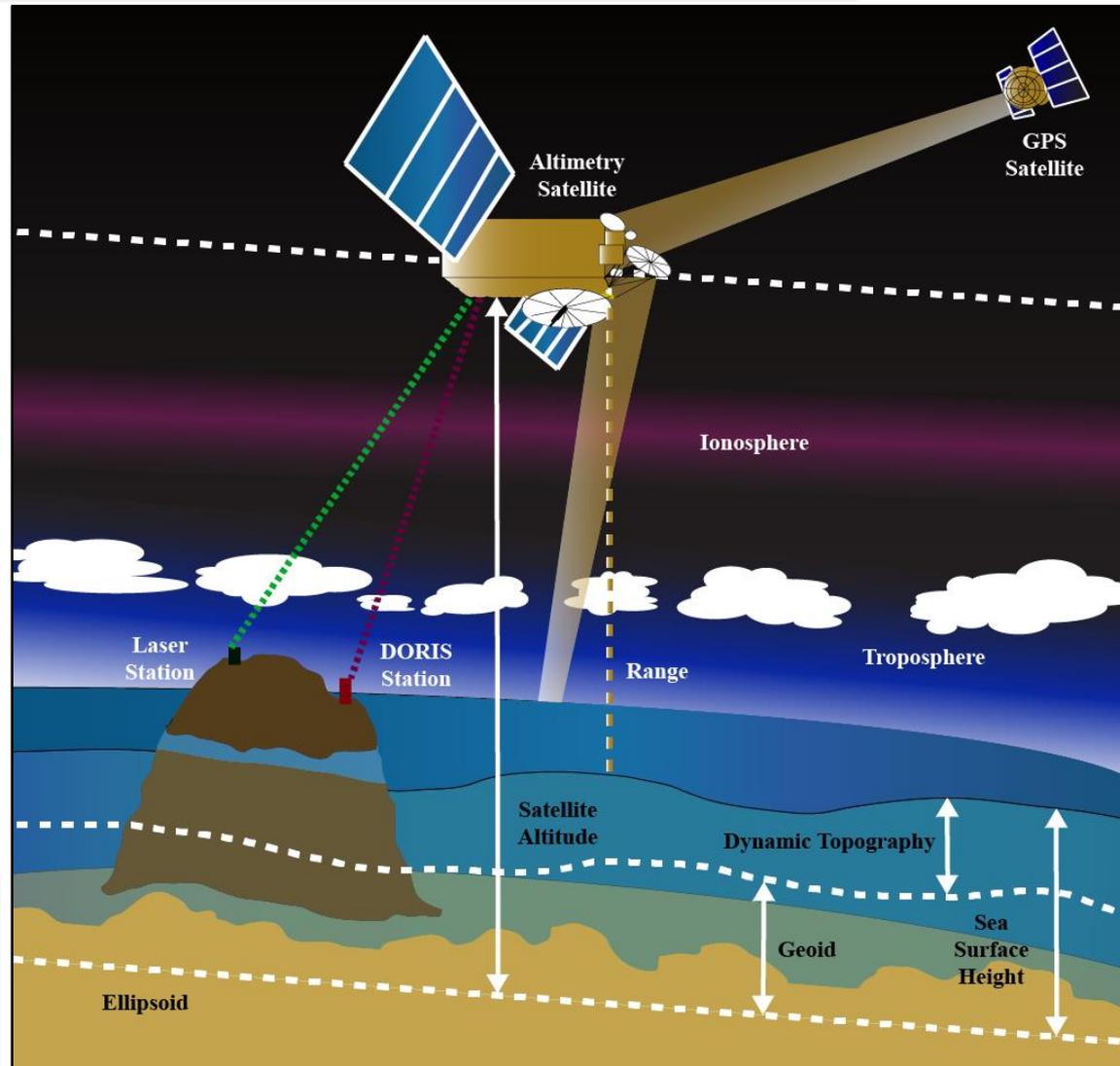
BENCH MARK 2 (1951) is a standard triangulation station disk, stamped "BLUFF TIDAL 1951", set in top of 2-inch iron pipe projecting 6 inches above ground to top of 2-inch soil pipe projecting 6 inches above ground, on point of land on north shore of Toksook Bay, on southern shore of western arm of Nelson Island and about 3 miles east-northeast of eskimo village of Unikumute. It is about 300 feet southwest of small stream, about 10 paces west of edge of 30-foot bluff and 300 feet northwest of triangulation station BLUFF (Bench Mark 2). Elevation: 38.16 feet above mean lower low water.



Data - Satellite altimetry

- Satellite altimeters have been in orbit continuously since 1992
- Satellites include GEOSAT, ERS-1 and ERS-2, TOPEX/Poseidon, GFO, Jason-1 and Jason-2, Envisat, CryoSat-2, SARAL, and HY-2A
- Data are processed and made available online through a range of agencies
- Variety of products available, such as mean sea level (MSL), wave height, sea level anomaly, atmospheric corrections and many more
- Unreliable near shore (30-50 km), noise from tides, land, and waves.

Figure 22. Satellite altimetry diagram showing instruments involved and measurement relationship to dynamic topography, the geoid, and ellipsoid.



Data - Satellite altimetry

- Data obtained from AVISO
- Gridded merged satellite dataset from 1992 – 2014 of MSL rates on $0.25^\circ \times 0.25^\circ$ cells

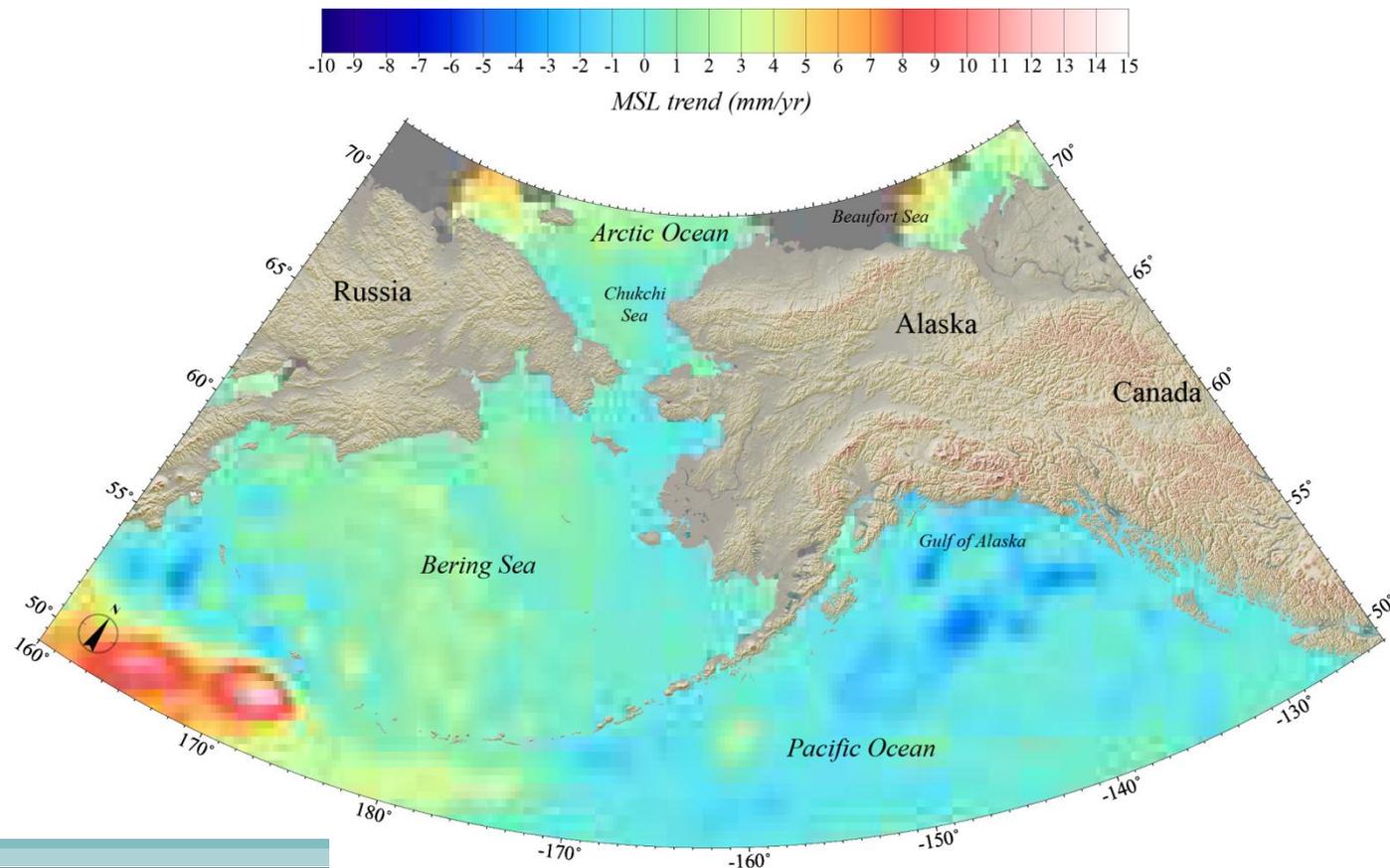


Figure 23. Satellite altimetry measurements of MSL rates over 22 years in the Bering Sea region. Data from all available satellites is merged and estimated on a $0.25^\circ \times 0.25^\circ$ grid

Data - Satellite altimetry

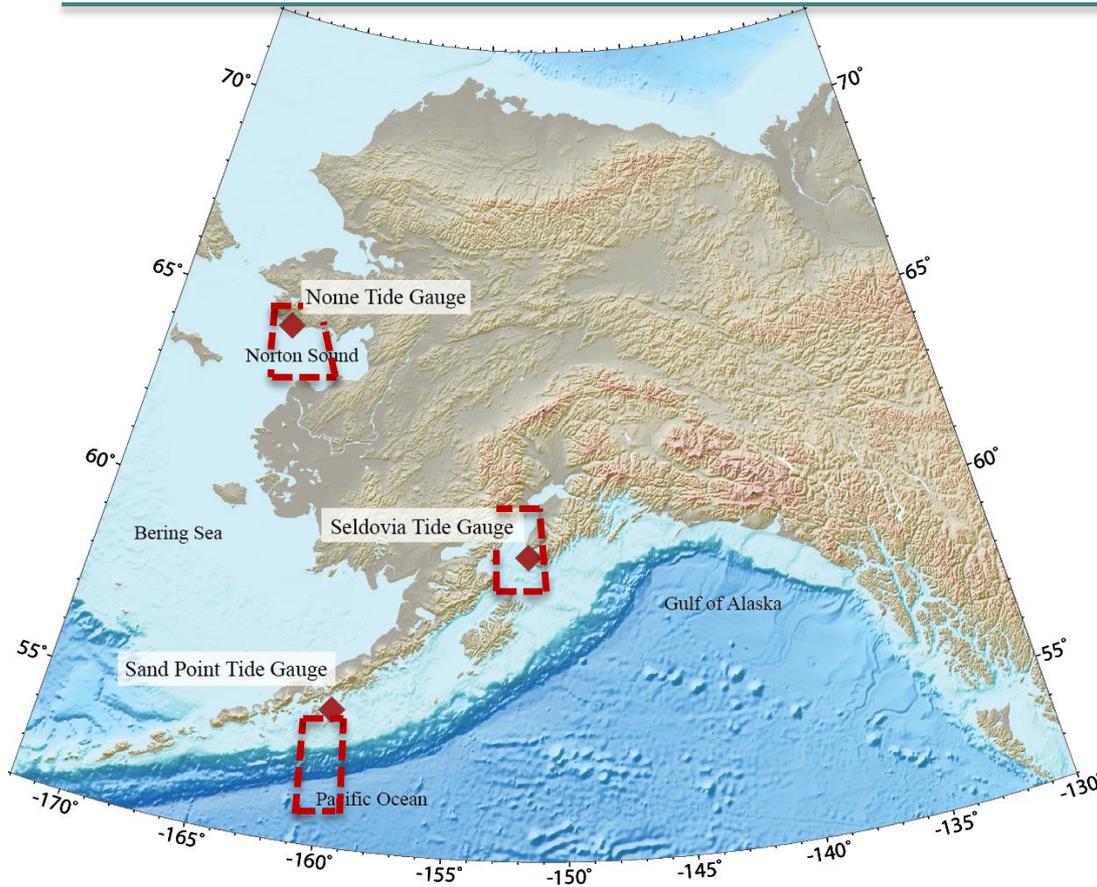
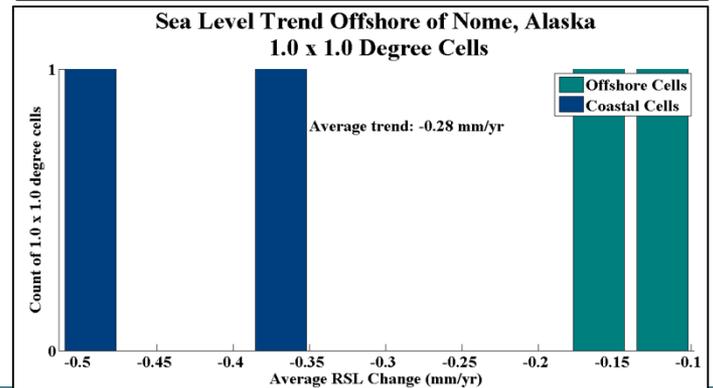
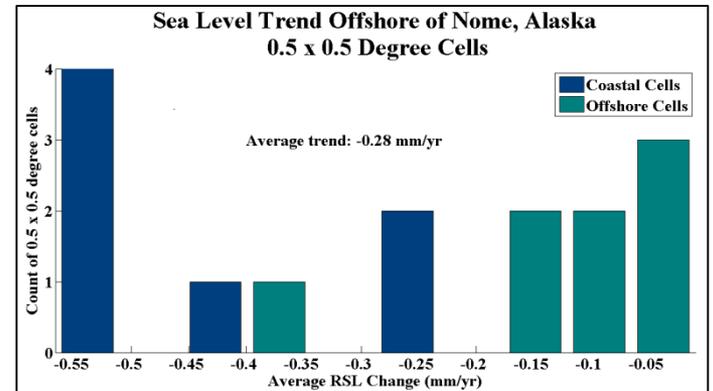
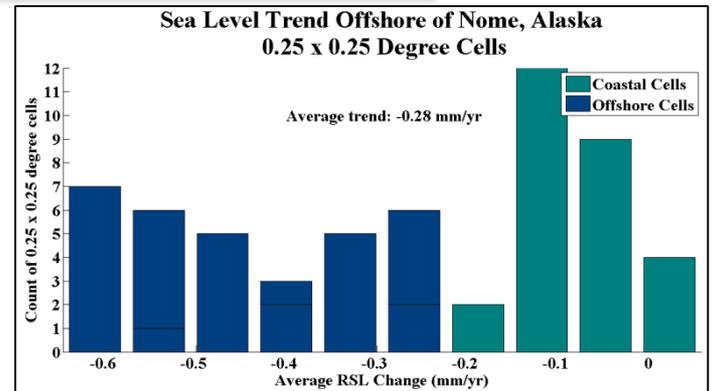


Figure 24. Satellite altimetry measurements of MSL rates over 22 years in the Bering Sea region. Data from all available satellites is merged and estimated on a $0.25^\circ \times 0.25^\circ$ grid



Data - Satellite altimetry

<i>Size of Averaged Cell Squares</i>	Nome		Sand Point		Seldovia	
	<i>Average MSL Rate (mm/yr)</i>	<i>Average Standard Deviation (mm/yr)</i>	<i>Average MSL Rate (mm/yr)</i>	<i>Average Standard Deviation (mm/yr)</i>	<i>Average MSL Rate (mm/yr)</i>	<i>Average Standard Deviation (mm/yr)</i>
0.25° x 0.25° cells		0.2054		0.8747		0.5609
0.50° x 0.50° cells	-0.2811	0.0554	1.1400	0.2529	0.1850	0.2254
1.00° x 1.00° cells		0.1077		0.4196		0.3556

- 0.5° x 0.5° averaged cells have lowest average standard deviation
- Minimized bias caused by coastal or offshore processes
- MSL rates used in this study will be these MSL rate values obtained from AVISO that are then averaged to 0.5° x 0.5° cells offshore of each community

Data - GPS

Global Positioning System – Campaign Stations



Photo courtesy of Jeff Freymueller

- Portable Surveying Equipment
- Precision of a few millimeters in 3D
- Repeated surveys measure motion of sites
- Occupations of 1-10 days

Figure 25. Tripod set up in Golovin, AK (left) and close up of spike mount occupation in Katmai (right).

Data - GPS

Global Positioning System – Continuous Stations



Figure 26. Continuous station with radon dome and solar and wind power.

Photo courtesy of Jeff Freymueller

Data - GPS

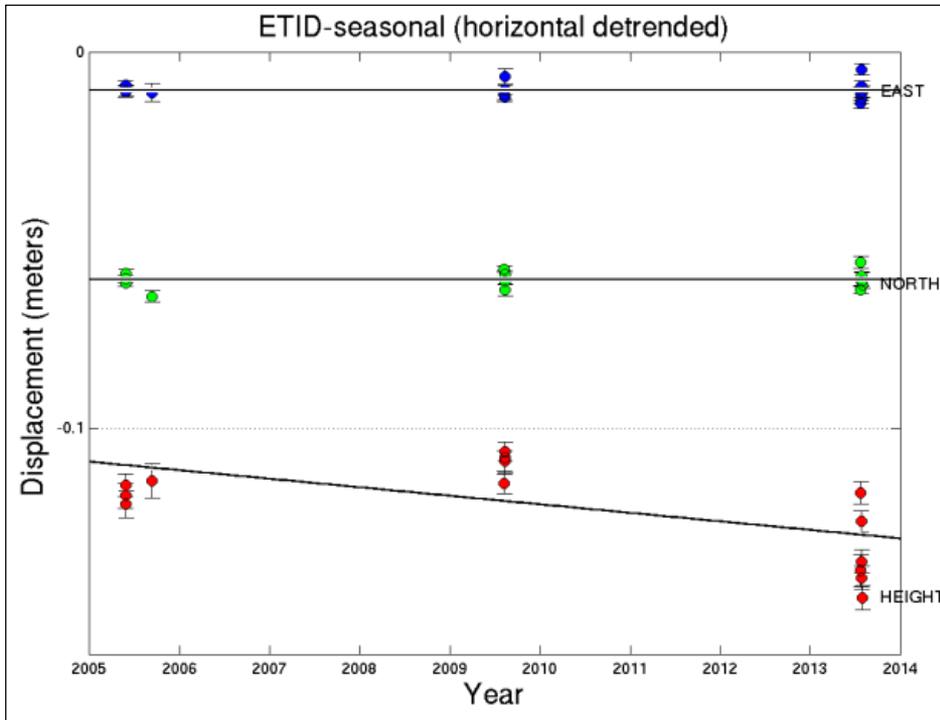


Figure 27. Campaign site (ETID) in Elim, AK with three occupations of multiple days each.

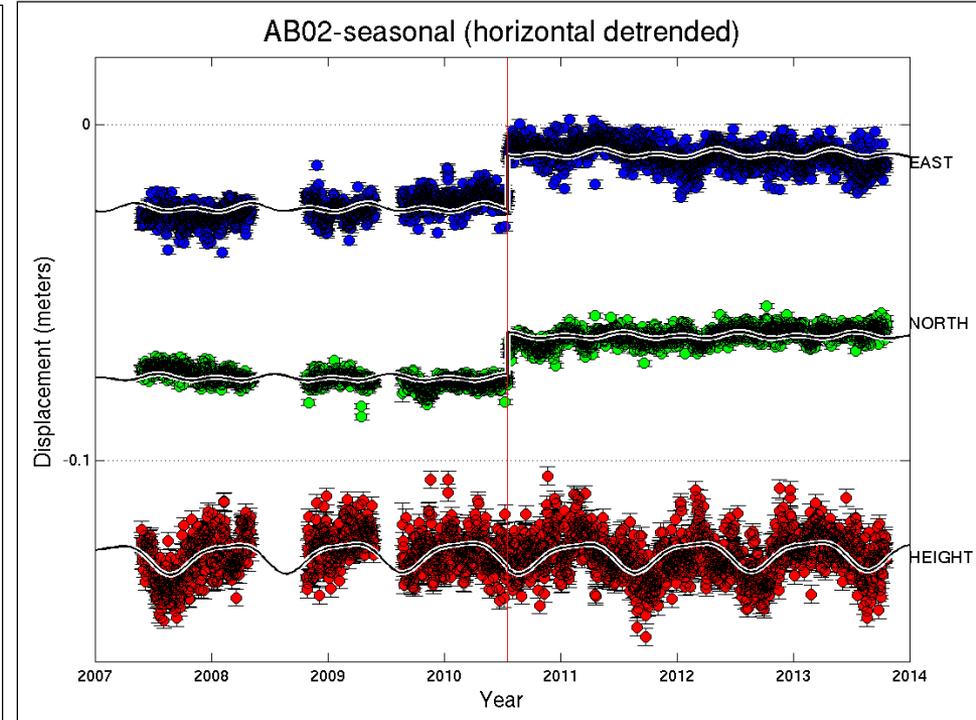


Figure 28. Continuous site (AB02) on Fox Island in the Aleutians. Data has been collected since 2007 and includes offset from an earthquake in 2010.

Data - GPS

- Velocity models fit to 864 GPS timeseries
- Combination of campaign and continuous GPS sites
- Lack of coverage in Western Alaska

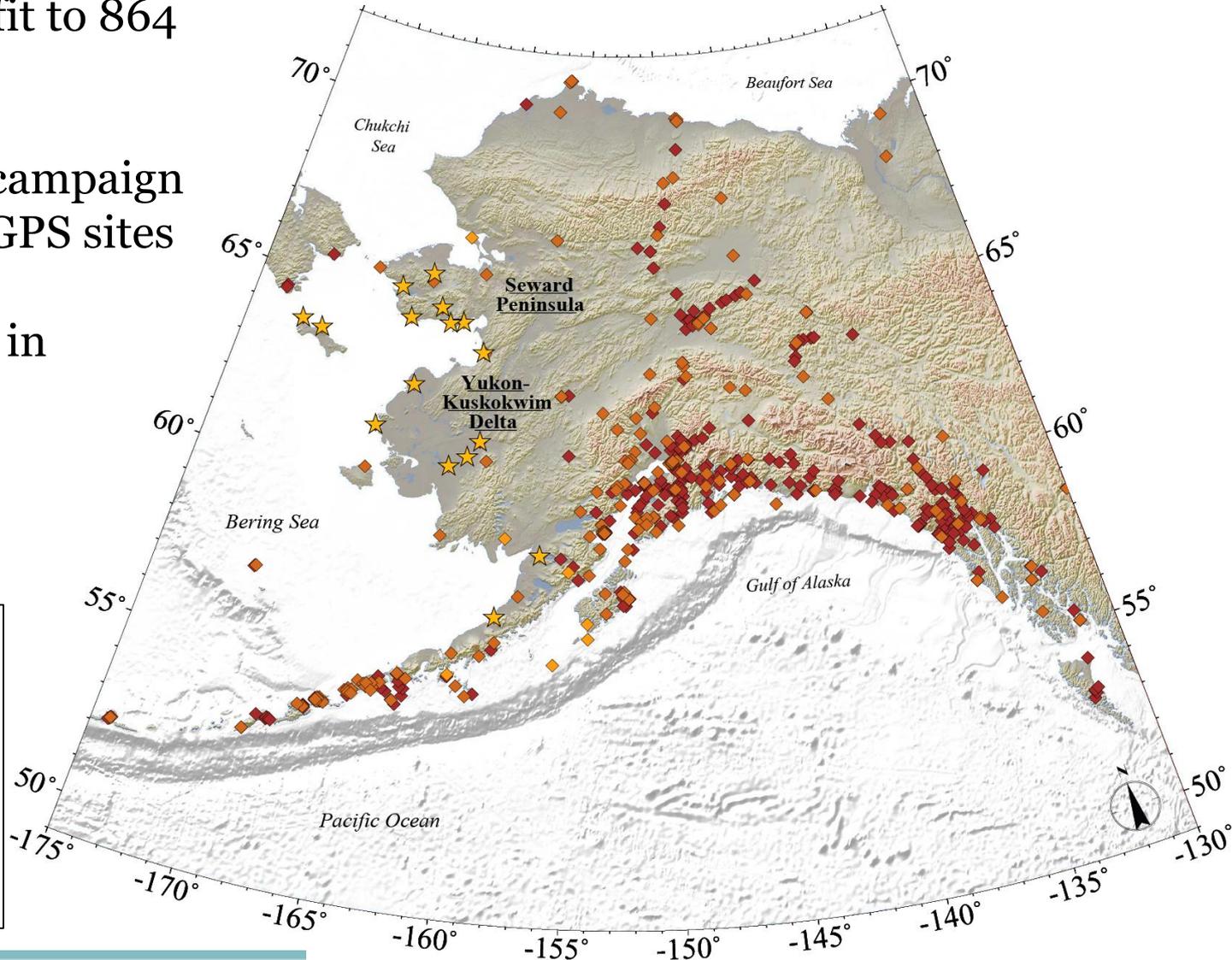
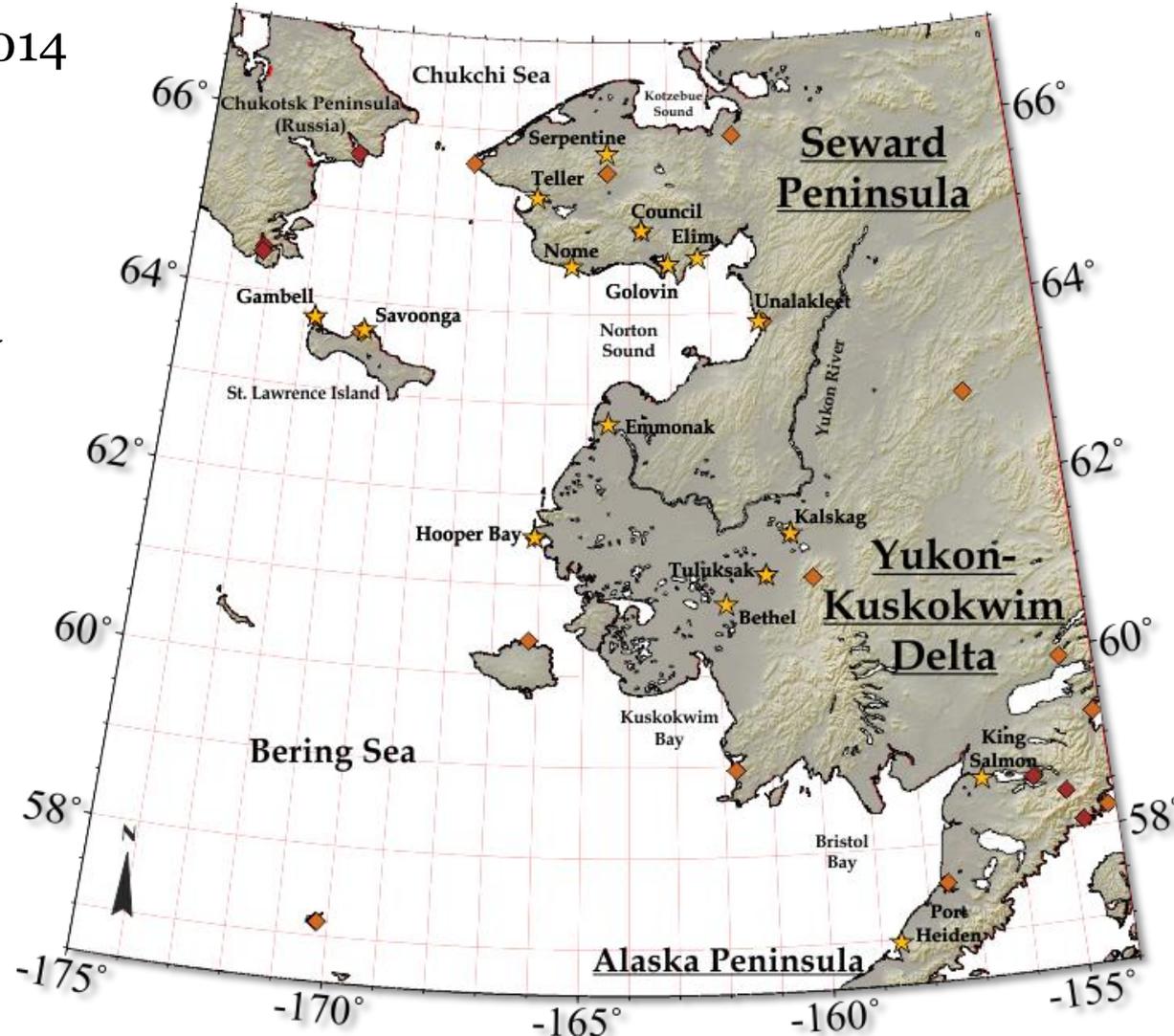


Figure 29. GPS sites in Alaska. Red diamonds are campaign sites, orange diamonds are continuous sites, and yellow stars are locations visited as part of this study.

Data - GPS

- Field work in 2013 and 2014
- 44 campaign sites in 16 communities
- Data was made publically available through NGS OPUS solutions database

Figure 30. Map of Western Alaska. GPS data for each of the mapped locations is stored in the UAF – GI geodetic database. Yellow stars are locations where campaign GPS surveys were conducted as part of this study. Red diamonds are campaign GPS sites and orange diamonds are continuous GPS sites



Data - Summary

- **Tide gauge** data are limited to Nome, and multiple datasets create ambiguity in measured rate. Will be used to “close the circle”.
- **Historic water level** data give inconsistent rates across Western Alaska and are suspected to have measurement errors. This data is not used further, but results are collected and available in my thesis.
- **Satellite Altimetry** data are available, but not ideal for near shore estimates. New method is developed here to minimize noise from coastal and oceanographic effects. Will be used to produce the MSL rate for Western Alaska.
- **GPS** data area available, but do not have good temporal or spatial coverage. Will be used to fit a GIA model to Western Alaska.

Results – Satellite altimetry

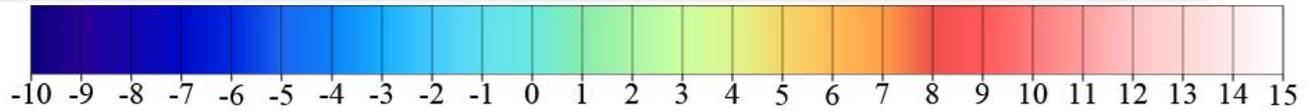
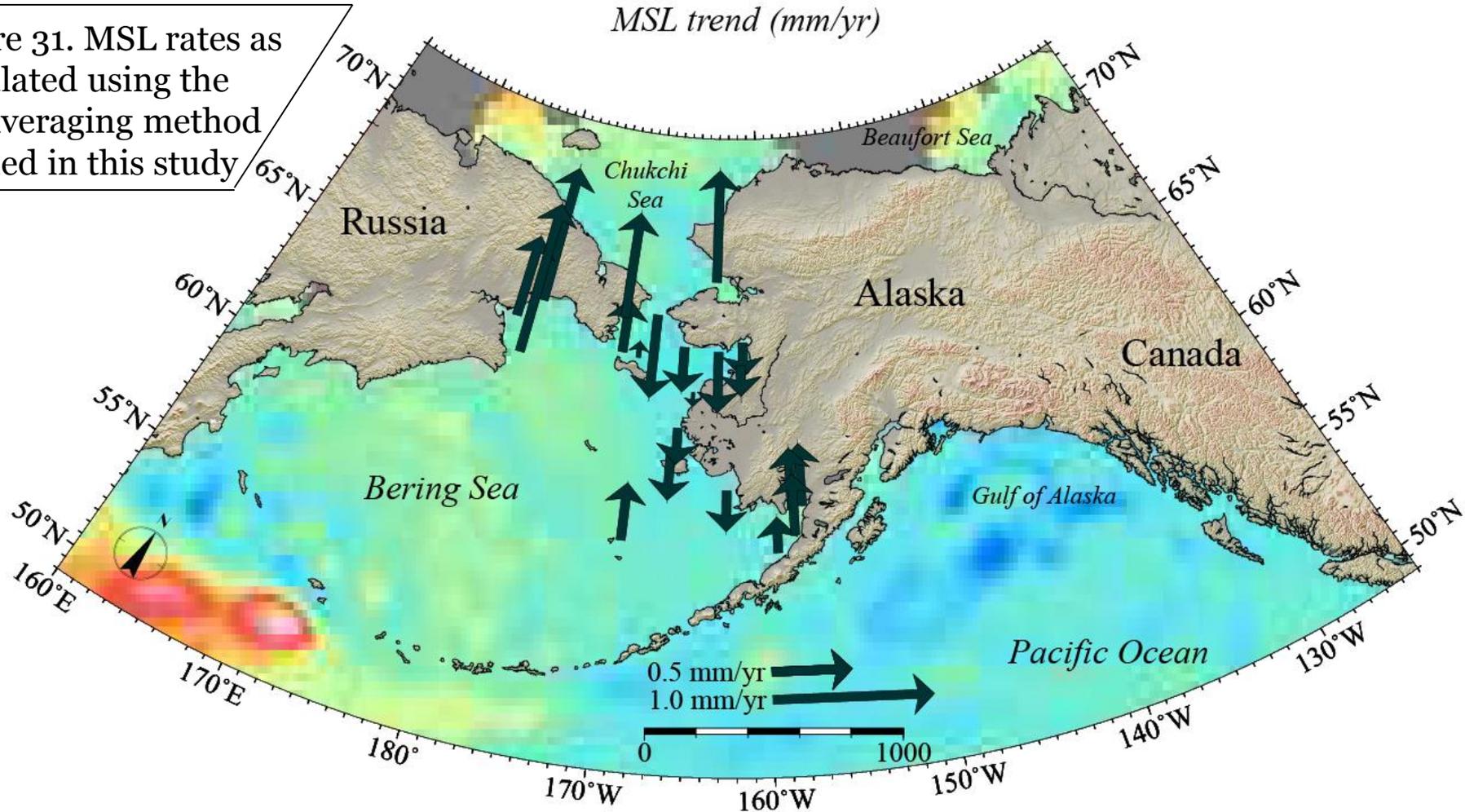


Figure 31. MSL rates as calculated using the cell averaging method defined in this study



Results – Satellite altimetry

- Visually Western Alaska has between -1 and +1 mm/yr of MSL change
- Using cell averaging method Western Alaska has an average MSL rate of -0.27 mm/yr
- Most of Western Alaska has a decrease in MSL, with increases in Bristol Bay, Kotzebue Sound, and the Gulf of Anadyr

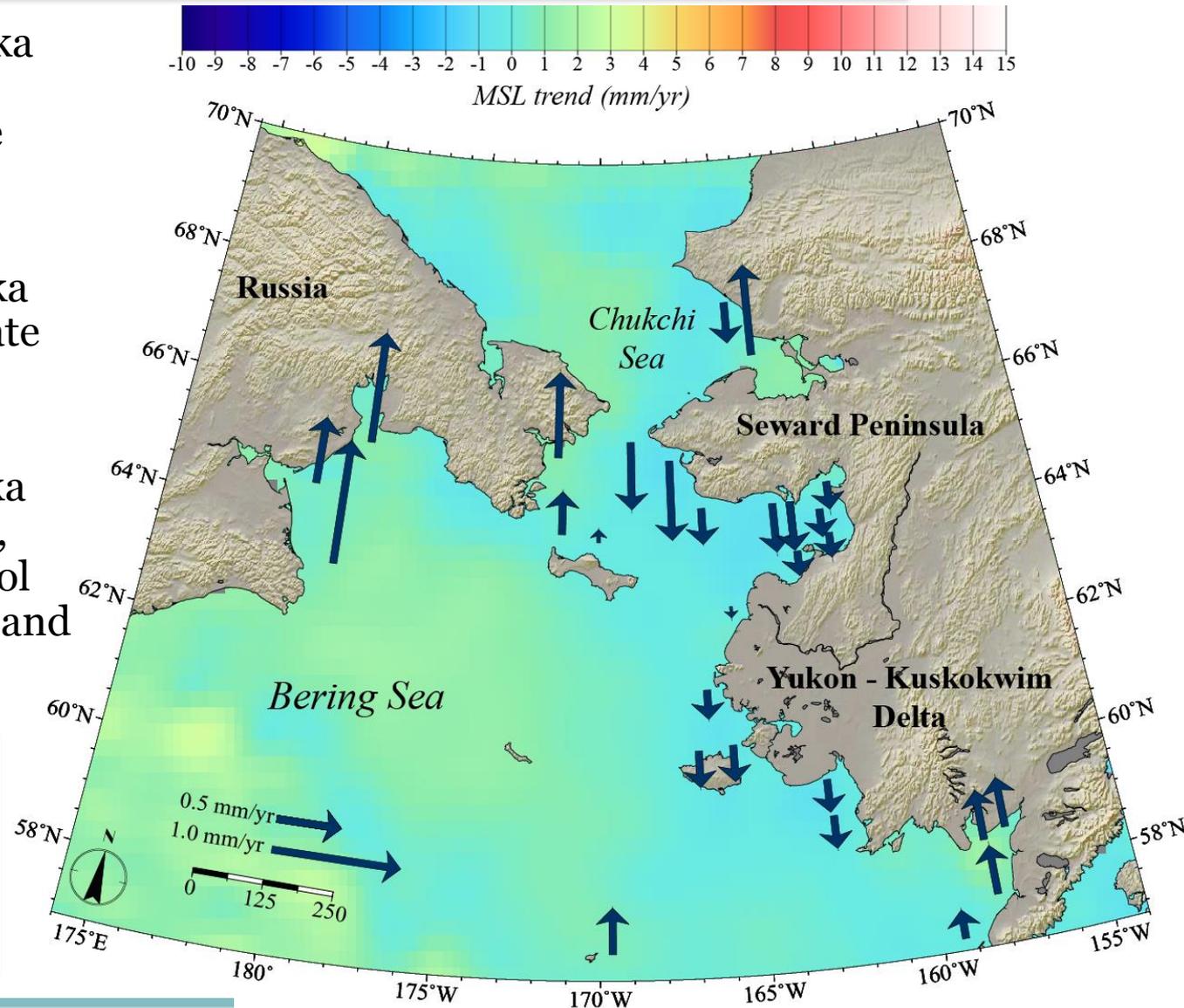


Figure 32. Map of gridded $0.25^\circ \times 0.25^\circ$ satellite altimetry cells with cell averaged rates imposed on top.

Results - GPS

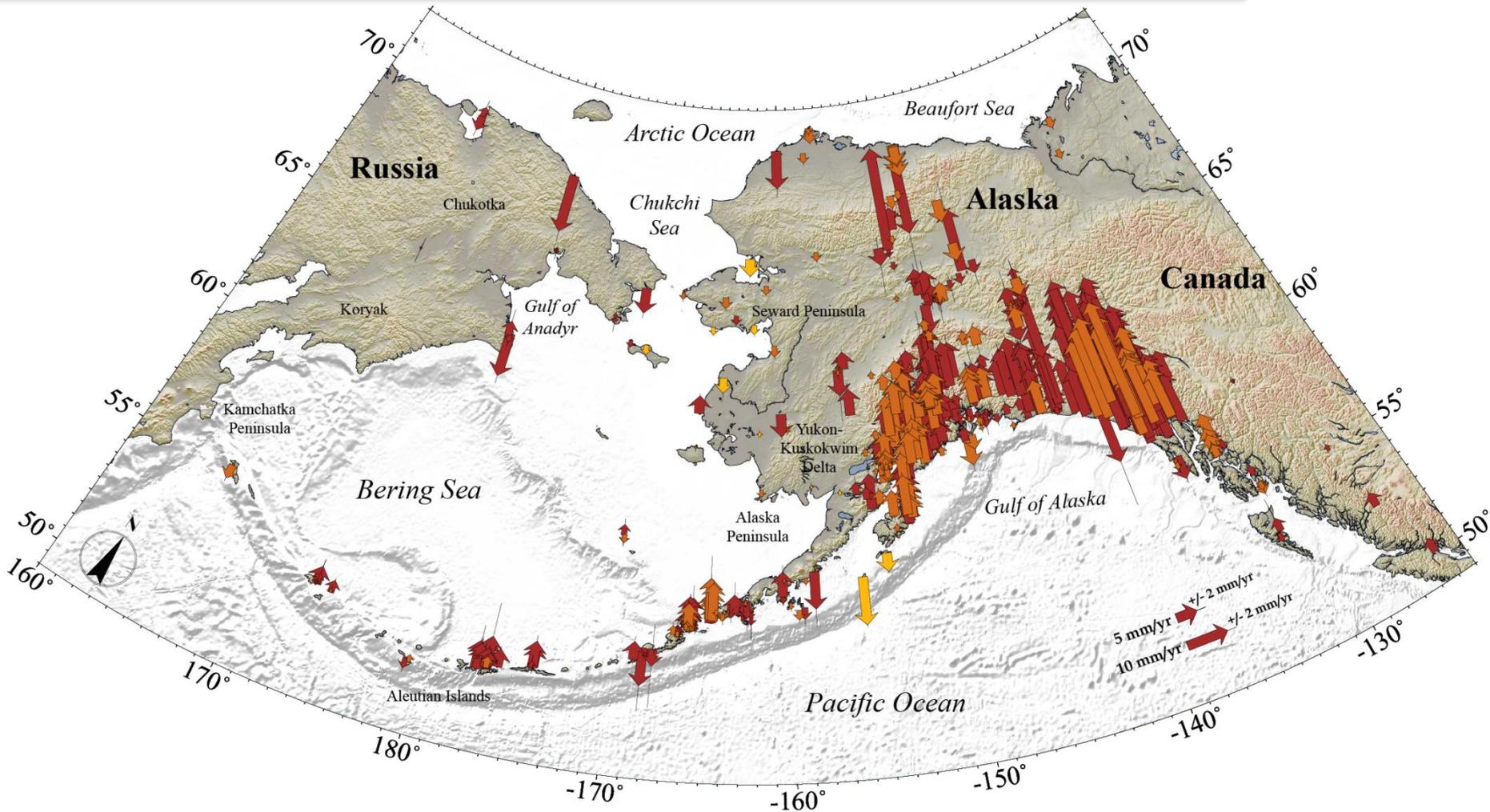
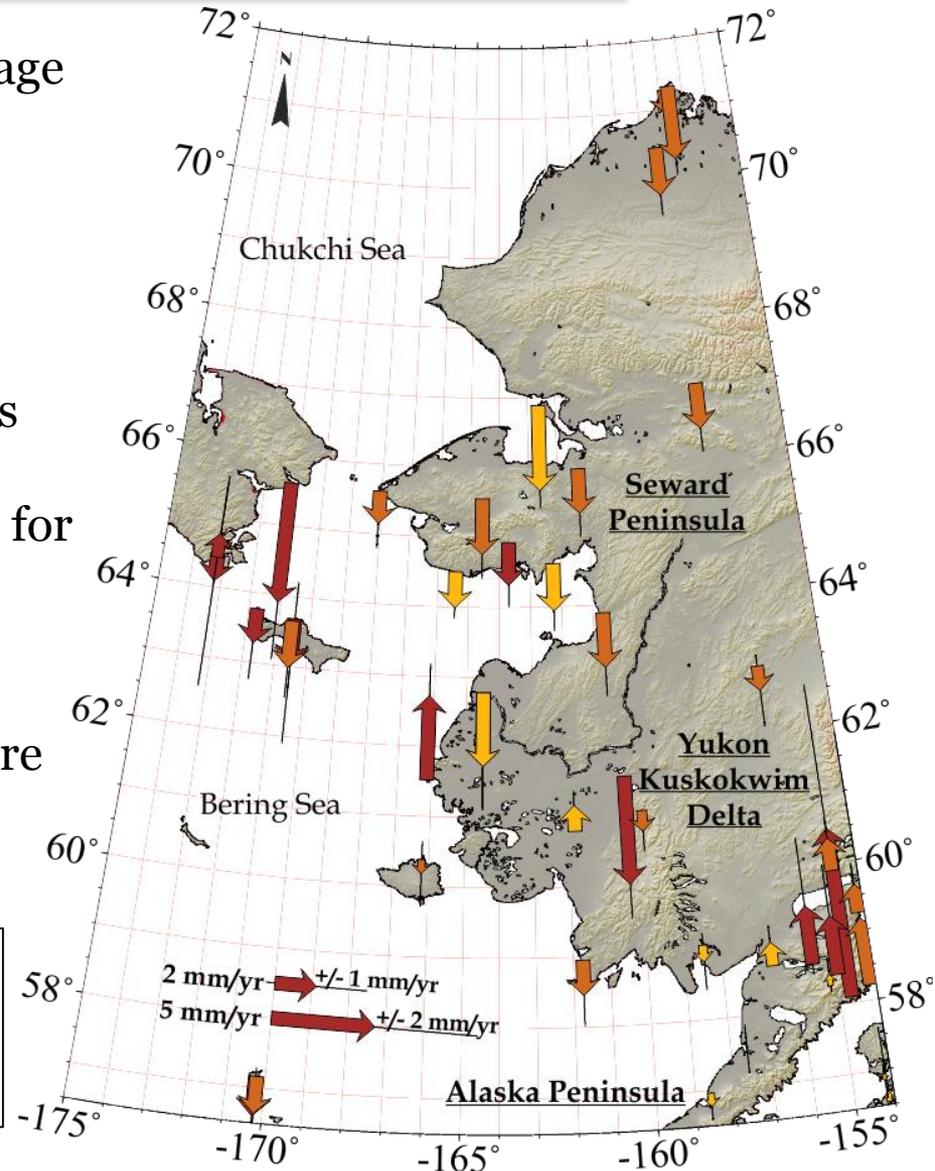


Figure 33. Map of GPS vertical velocities in Alaska. Red arrows are from campaign sites, orange arrows are continuous sites, and yellow arrows are where a weighted mean was taken of multiple GPS sites.

Results - GPS

- Vertical velocities in Western Alaska average to -1.41 mm/yr
- Subsidence supports GIA forebulge relaxation theory
- Seward Peninsula has consistent velocities showing subsidence, but the Yukon-Kuskokwim Delta shows a variety of rates for both uplift and subsidence
- Localized effects such as frost jacking, slumping, erosion, or human tampering are suspected at many sites in the Yukon-Kuskokwim Delta

Figure 34. Map of GPS vertical velocities in Western Alaska. Red arrows are from campaign sites, orange arrows are continuous sites, and yellow arrows are where a weighted mean was taken of multiple GPS sites.



Results - GIA

Model

TABOO – Post Glacial Rebound Calculator

Input: loading, earth model, and time of evaluation

Output: E,N, up velocities and displacements

Constants

Lithosphere + Asthenosphere + Upper Mantle = **670 km**

Lower mantle viscosity = **3.0×10^{21} Pa s**

Loading model = **ICE-3G**

Model Parameters

Lithosphere thickness: **25 – 120 km**

Asthenosphere thickness: **40 – 340 km**

Asthenosphere viscosity: **5.0×10^{18} - 2.0×10^{21} Pa s**

Upper mantel viscosity: **1.0×10^{20} - 2.0×10^{21} Pa s**

Results - GIA

Best Fit Model

Lithosphere thickness = **120 km**

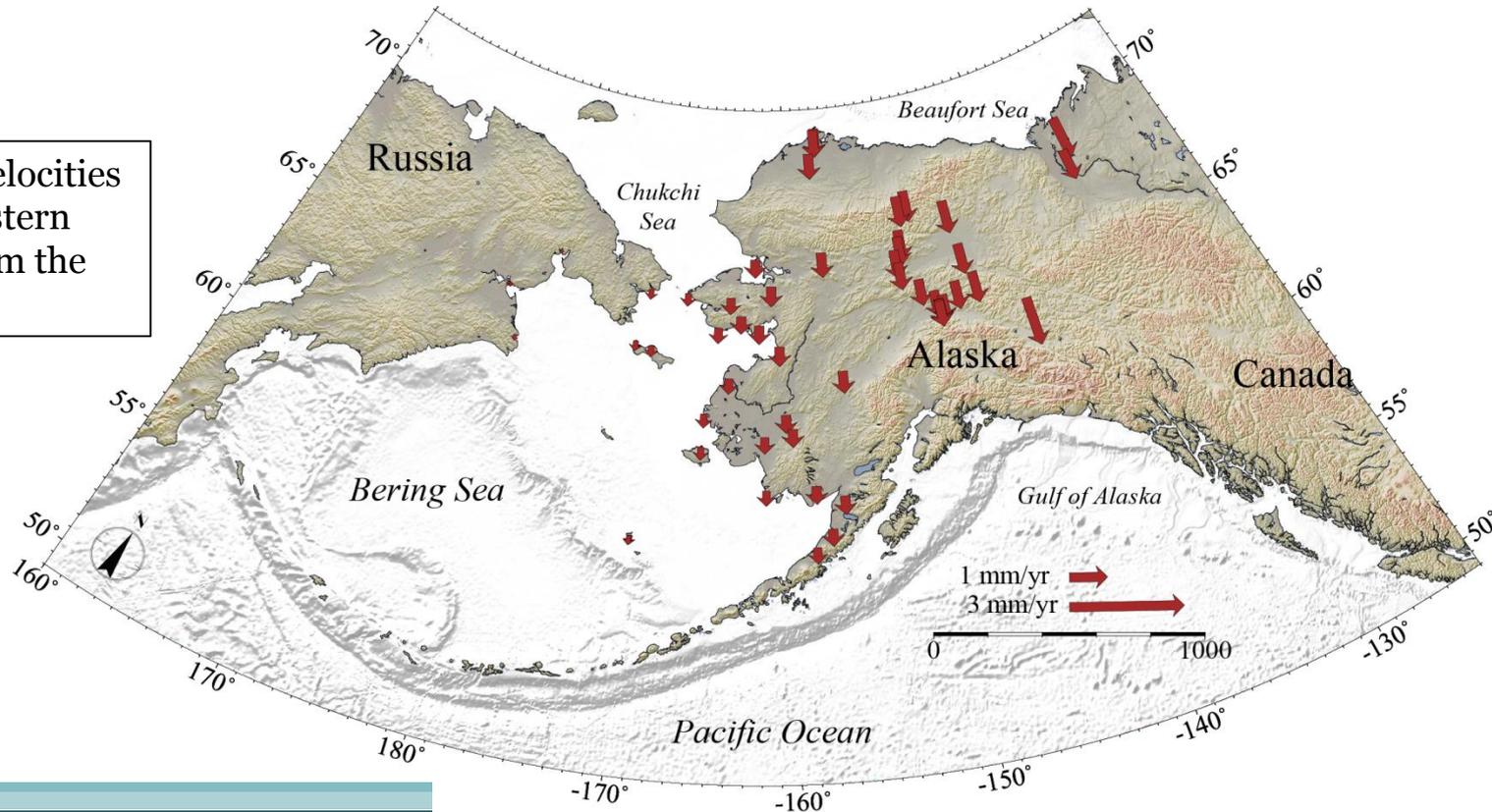
Asthenosphere thickness = **100 km**

Asthenosphere viscosity = **2.5×10^{19} Pa s**

Upper Mantle Viscosity = **1.5×10^{21} Pa s**

RMS calculated for each model using 50 GPS sites

Figure 35. Vertical velocities in Northern and Western Alaska estimated from the best fit GIA model



Results - GIA

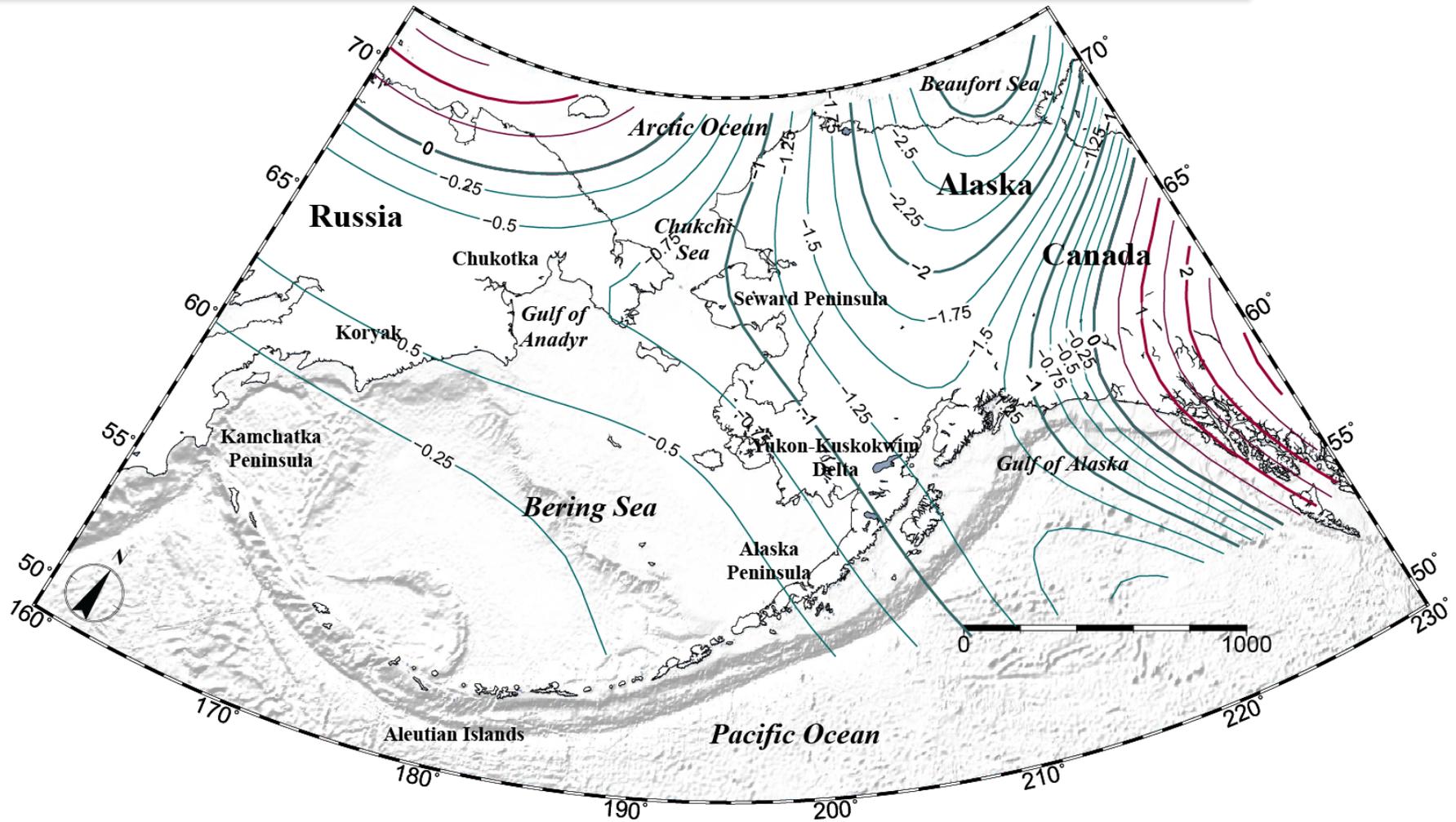
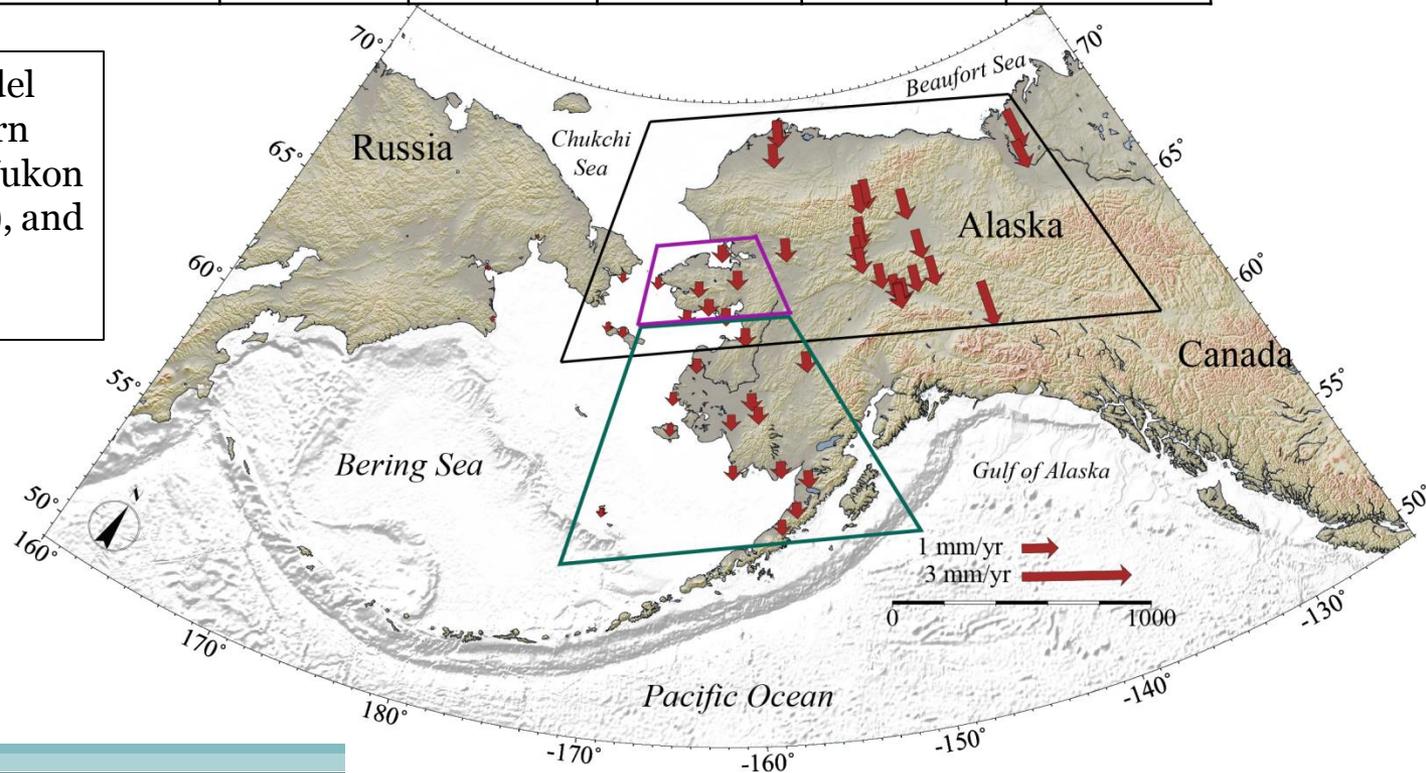


Figure 36. Contour plot of best fit GIA model. Red lines are uplift and green are subsidence.

Results - GIA

Best fit parameters	Region Model is Fit to				
	All	North	West	Seward	Delta
Lithosphere Thickness (km)	120	120	80	30	120
Asthenosphere Thickness (km)	100	120	280	80	300
Asthenosphere Viscosity (10^{21} Pa s)	0.025	0.025	0.1	0.005	1
Upper Mantle Viscosity (10^{21} Pa s)	1.5	2	2	2	1
Normalized root mean square	0.1365	0.1312	0.1783	0.3510	0.1814

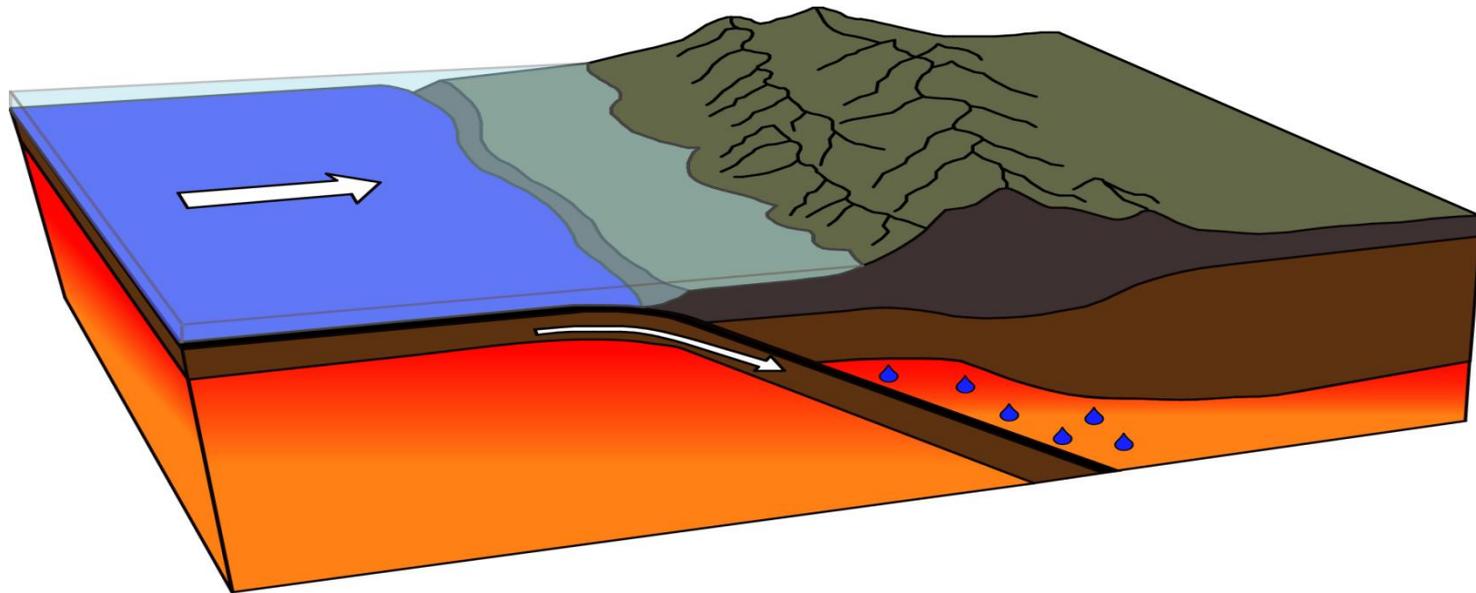
Figure 37. Best fit GIA model for Alaska with the Northern (black), Seward (purple), Yukon –Kuskokwim Delta (green), and Western (purple + green) regions outlined.



Results - GIA

- 120 km lithosphere = Continental root
- Four layer model is very consistent in Alaska, with clear asthenosphere/upper mantle boundary
- Low viscosity asthenosphere is more characteristic of hot, wet mantle resulting from active subduction in Southern Alaska
- Low viscosity lower mantle is better fit than a higher viscosity lower mantle for Northern and Western Alaska
- Model is a better representation of vertical velocity in Western Alaska

Figure 39. Profile of best fit earth models in Alaska with subduction zone and thinner (55 km) lithosphere in the South (left) and thicker (120 km) lithosphere to the North. This also shows the hot, wet asthenosphere that extends underneath Northern Alaska



Results - Closing the circle

$$\text{Satellite Altimetry (MSL trend)} - \text{GPS/GIA (tectonic vertical velocity)} = \text{Tide Gauge (RSL change)}$$

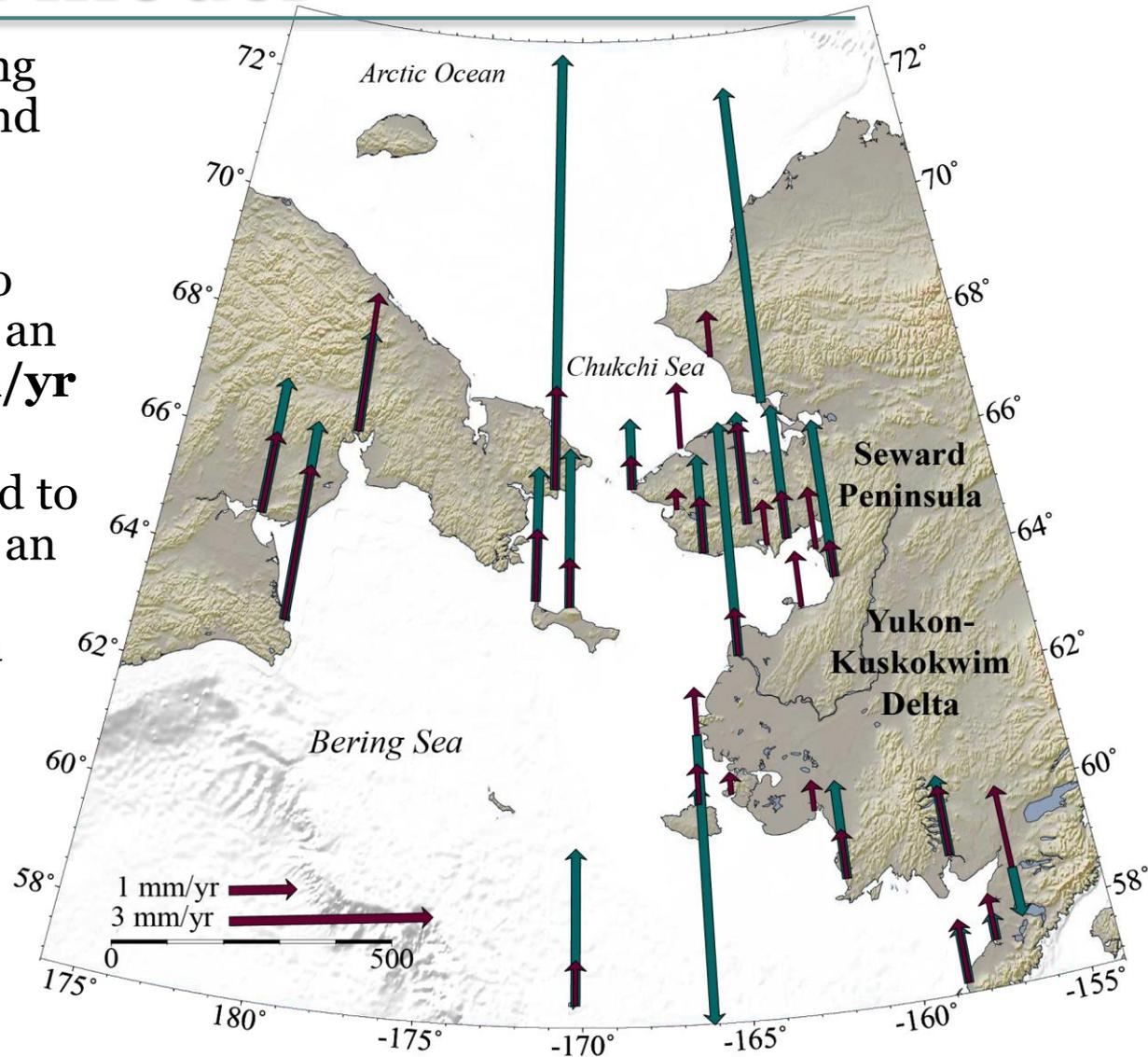
- Misclosure of the circle is -1.97 mm/yr (Sand Point) and -2.09 mm/yr (Seldovia)
- Additive error shows 1.59 mm/yr (Sand Point) and 1.80 mm/yr (Seldovia)
- Additive error suggests additional error of approx. 1.75 mm/yr unaccounted for
- Studies suggest satellite altimetry has additional regional error of 1-2 mm/yr

	Sand Point		Seldovia	
	Velocity (mm/yr)	Uncertainty (+/- mm/yr)	Velocity (mm/yr)	Uncertainty (+/- mm/yr)
MSL (satellite altimetry)	1.69	0.60	0.21	0.60
Tectonic velocity (GPS)	-1.12	0.30	7.47	0.30
RSL calculated (MSL - tectonic velocity)	2.81	0.90	-7.26	0.89
RSL observed (tide gauge)	0.84	0.95	-9.35	0.82
RSL misfit (observed - calculated)	-1.97		-2.09	
Root sum of squares of all uncertainties	1.16		1.06	
Additive error of misfit and RSS of uncertainties	1.59		1.80	

Results - RSL model

- RSL change calculated using satellite altimetry values and GPS (green) or GIA (red) tectonic vertical velocities
- GPS measurements used to calculate RSL change have an average rate of **+1.20 mm/yr** in Western Alaska
- GIA model predictions used to calculate RSL change have an average rate of **+0.84 mm/yr** in Western Alaska
- Over all of Western Alaska RSL is increasing

Figure 40. RSL change map for Western Alaska. Green arrows are RSL change calculated with GPS measurements and red arrows are RSL change calculated using best fit GIA model.



Results - RSL model

- Average RSL change on the Seward Peninsula is **+0.89 mm/yr**
- RSL change calculated using best fit GIA model everywhere except for Kotzebue
- Kotzebue determined to have local effect occurring that is not a result of GIA

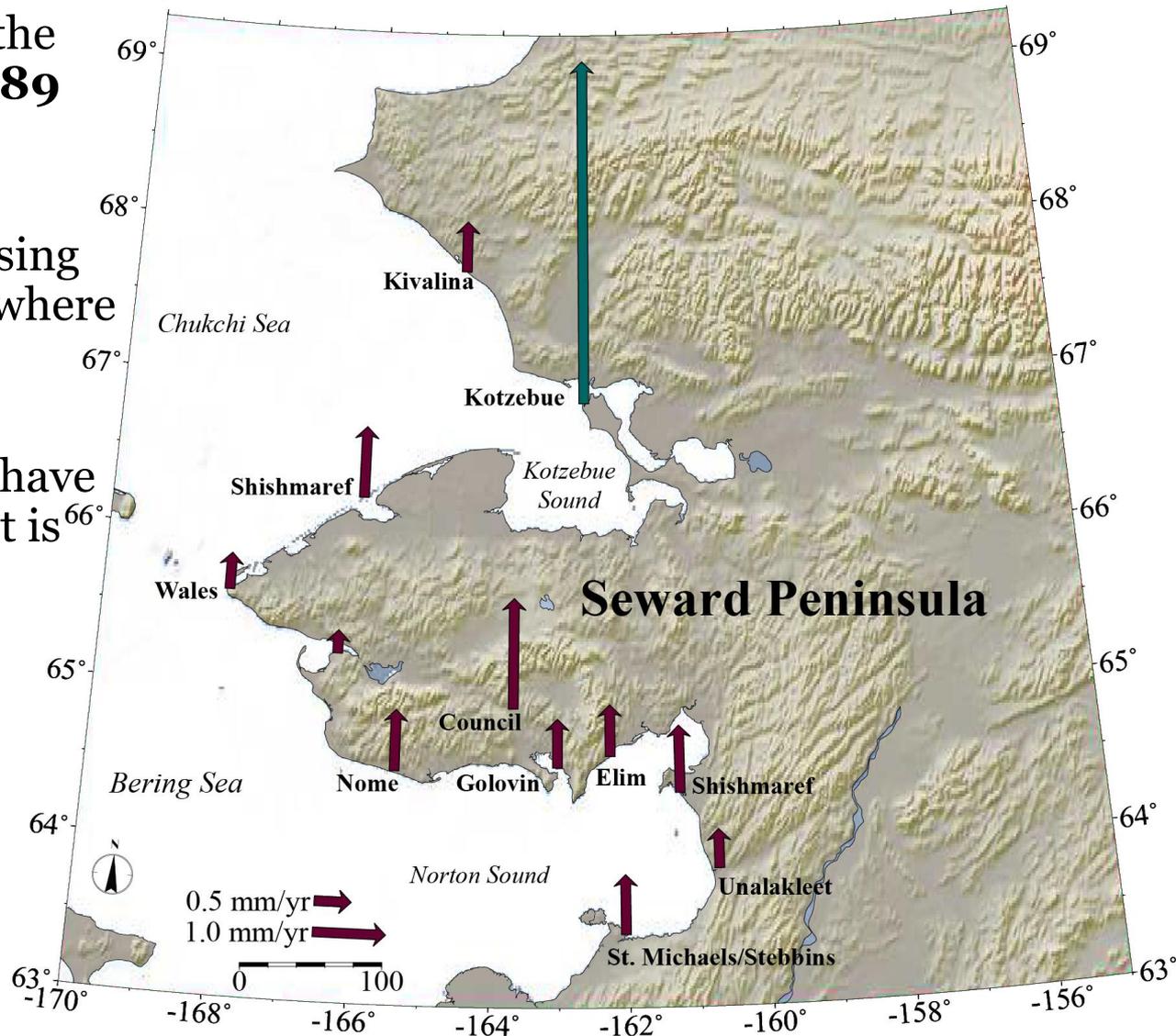


Figure 41. RSL change map for the Seward Peninsula. Green arrows are RSL change calculated with GPS measurements and red arrows are RSL change calculated using best fit GIA model.

Results - RSL model

- Average RSL change on the Yukon-Kuskokwim Delta is **+0.79 mm/yr**

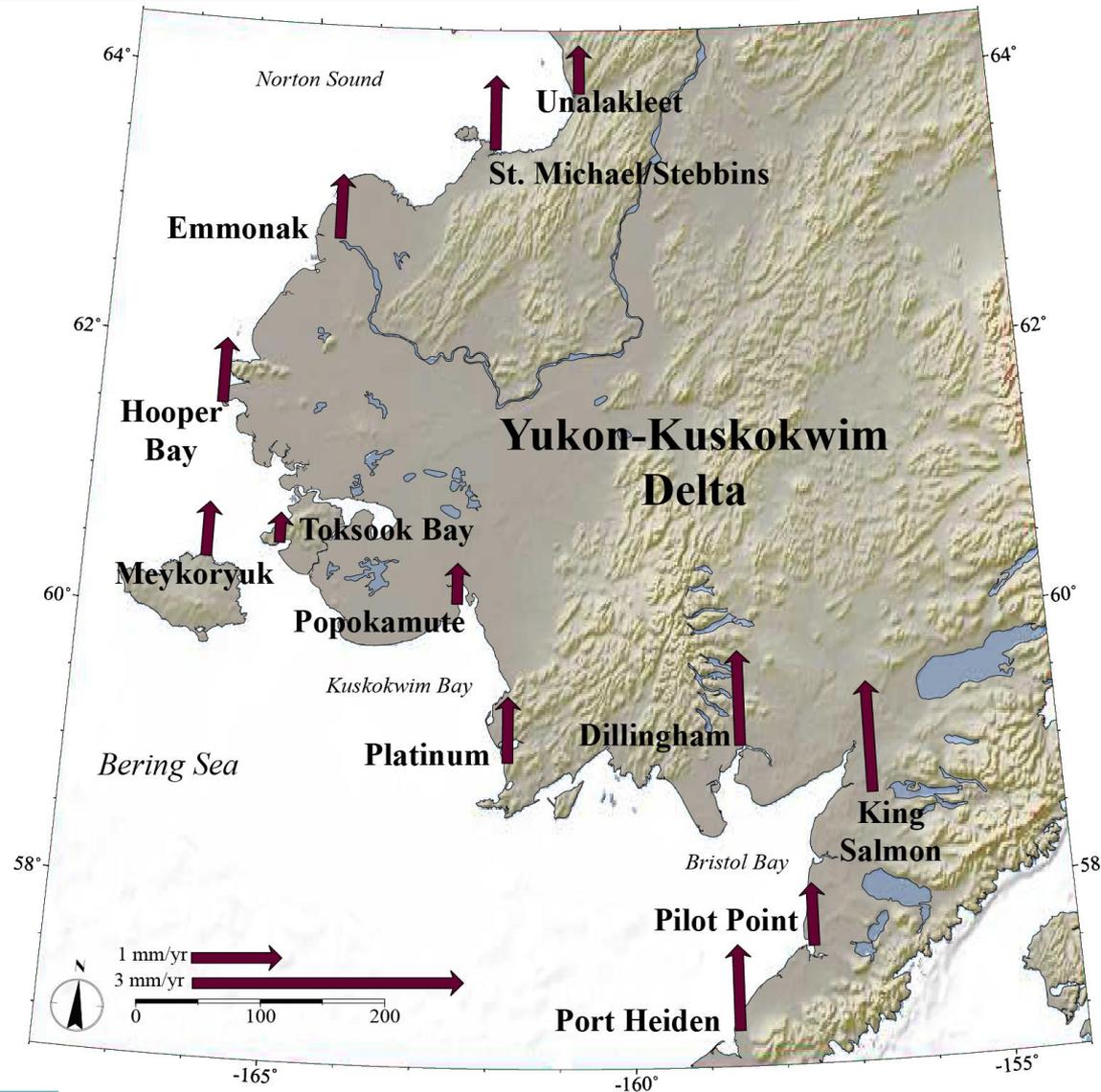


Figure 42. RSL change map for the Yukon-Kuskokwim Delta. Green arrows are RSL change calculated with GPS measurements and red arrows are RSL change calculated using best fit GIA model.

Summary

- **Nome tide gauge** estimates **+0.50 mm/yr** RSL change, but there is a wide range depending on which dataset is used.
- **Satellite altimetry** using averaging method developed here estimates MSL change for Western Alaska to be **-0.27 mm/yr**.
- **GPS** tectonic vertical velocities for Western Alaska average to **-1.41 mm/yr**.
- Best fit **GIA model** predicts **-1.06 mm/yr** for tectonic vertical velocity.
- **RSL change model** using satellite altimetry and best fit GIA model estimates **+0.84 mm/yr**.
- Best fit earth model parameters for Northern and Western Alaska are a continental shield type lithosphere (120 km) with an intermediately thick asthenosphere (100 km) that has a viscosity of 2.5×10^{19} Pa s and an upper mantle viscosity of 1.5×10^{21} Pa s.

Summary - Products of This Study

Collection of tide gauge and water level data into one source

Analysis of tide gauge data available from multiple sources (Appendix 2)

Development of satellite altimetry averaging of cells method for use in near shore applications (Appendix 5)

Augmentation of the GPS network in Western Alaska (Appendix 3)

Updated vertical and horizontal velocity model for Alaska (Appendix 4)

GIA model for Northern and Western Alaska

Earth model better defined for Northern and Western Alaska

RSL model for Western Alaska



Future Work

- **DATA! NEED MORE DATA! GET DATA! DATADATADATA**
 - Continue collection of GPS data in Western Alaska – Appendix 9 in Thesis gives detailed report for prioritization
 - Tide gauge installations needed.... Everywhere but Nome...
- Implementation of new satellite altimetry algorithms supposed to be available October, 2015 (Ablain et al., 2015).
- Development of YK Delta subsidence model
 - Need DATA: deposition area and rate as well sediment type/load.
 - Could use ice loading model with load changed to sediment on the delta instead of ice on the continent.
- Updating and further analysis of the RSL model, GIA model, and satellite altimetry methods used in this study.

Thank You!



Figure 8. Elim “field assistants” with spike mount at site ETID in the Summer of 2013.

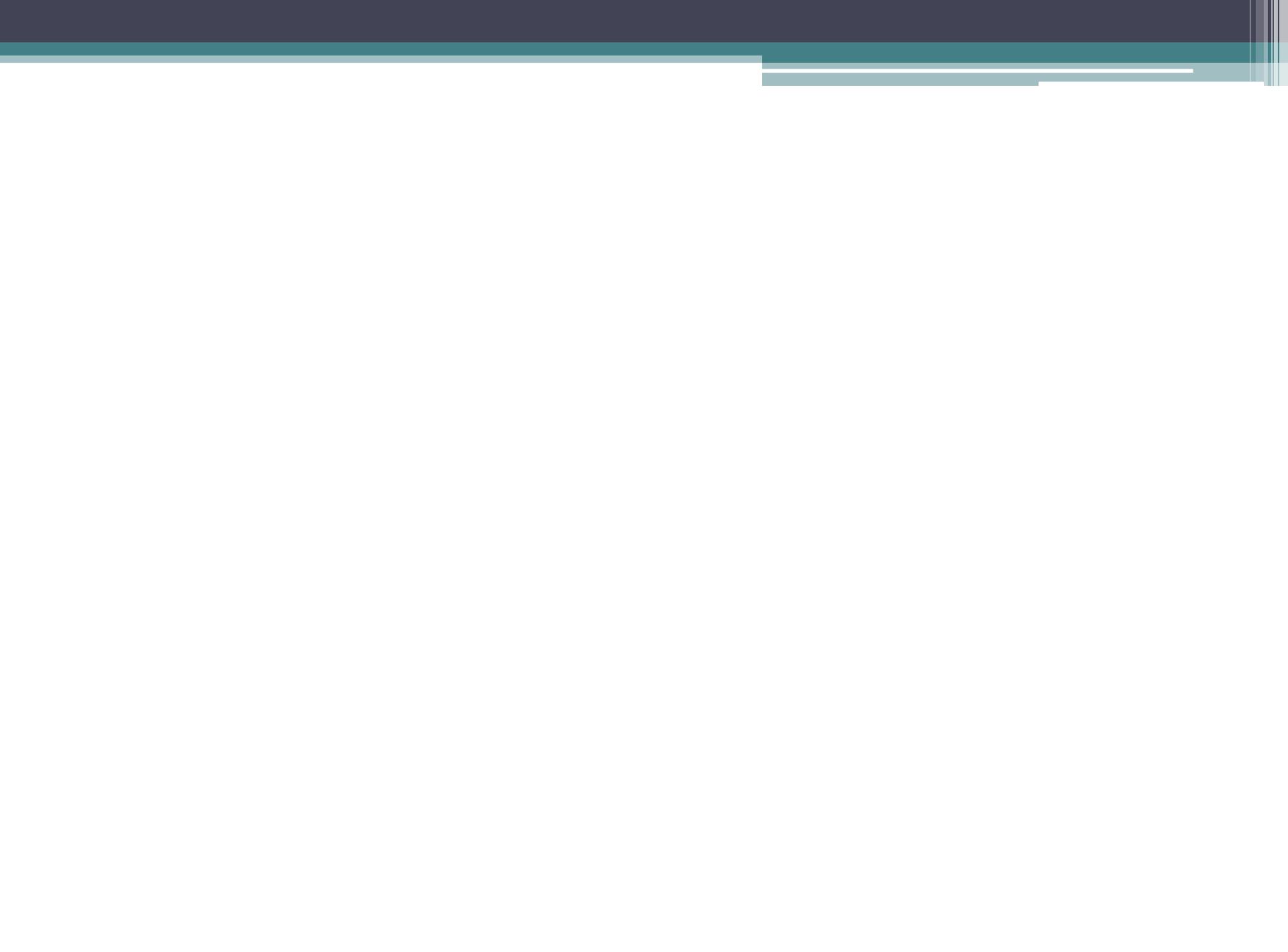
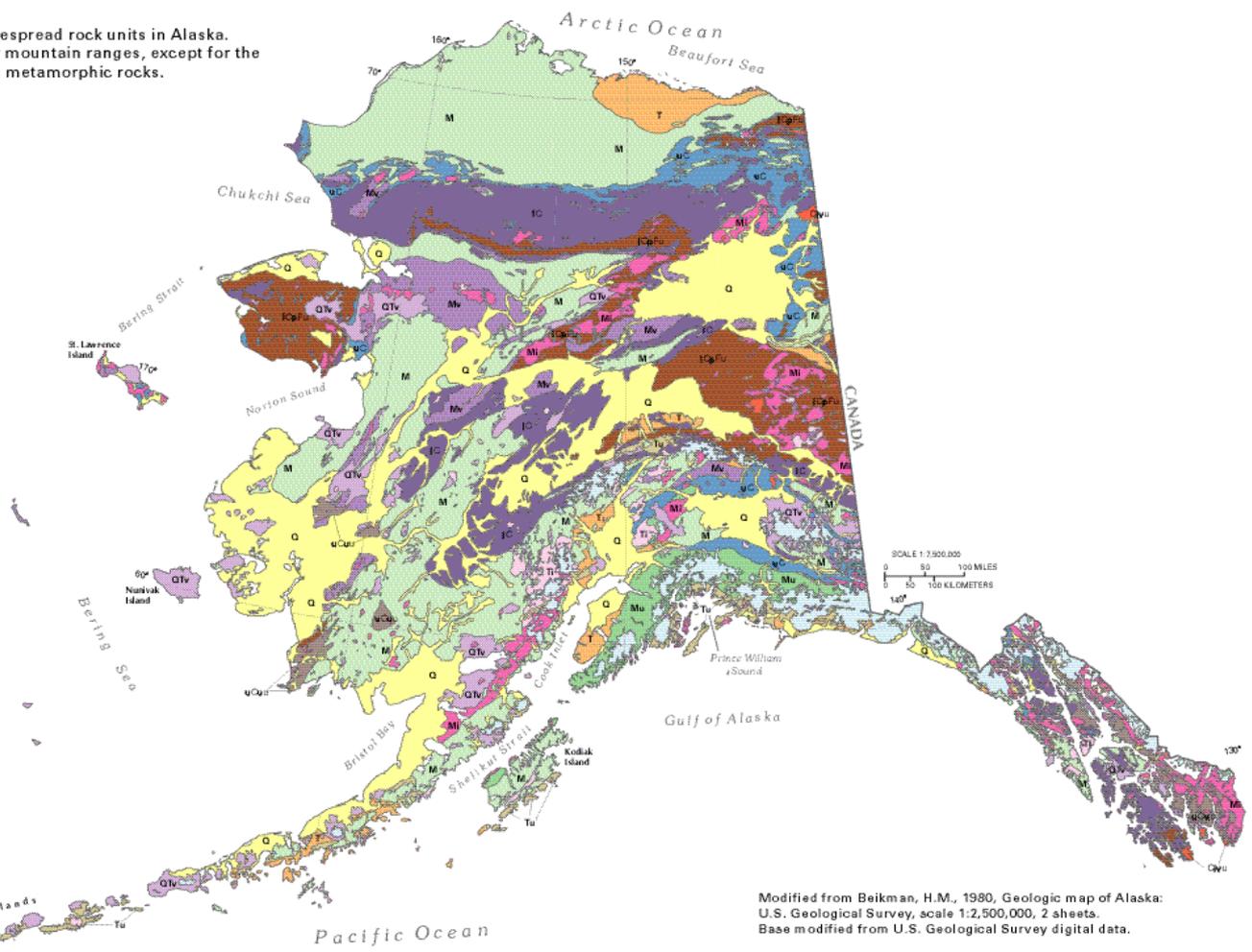


Figure 10. Paleozoic and Mesozoic sedimentary rocks are the most widespread rock units in Alaska. Volcanic and intrusive igneous rocks underlie most of the State's major mountain ranges, except for the Brooks Range, which is underlain mostly by Paleozoic sedimentary and metamorphic rocks.

EXPLANATION

Quaternary	Quaternary and Tertiary
Q Unconsolidated deposits	QTV Volcanic rocks
Tertiary	Tertiary
T Sedimentary rocks	Ti Intrusive igneous rocks
Mesozoic	Tu Volcanic and sedimentary rocks
M Sedimentary rocks	Mesozoic
Mississippian through Permian	Mv Volcanic rocks
uC Sedimentary rocks	Mi Intrusive igneous rocks
Cambrian through Devonian	Mu Metamorphic, volcanic, and igneous intrusive rocks
IC Sedimentary rocks	Upper Paleozoic
	uCu Metamorphic, sedimentary, and igneous rocks
	Paleozoic
	Clvu Igneous intrusive and volcanic rocks
	Lower Paleozoic and Precambrian
	ICpPv Metamorphic rocks
	Glacier
	No data
	Fault —Dashed where approximately located



Modified from Beikman, H.M., 1980, Geologic map of Alaska: U.S. Geological Survey, scale 1:2,500,000, 2 sheets. Base modified from U.S. Geological Survey digital data.

$$RMSN = \sqrt{\frac{\sum_{i=1}^n w(i)(v(i)_{gps} - v(i)_{gia})}{\sum_{i=1}^n w(i)}}, \quad w(i) = \frac{1}{\sigma(i)^2}$$

where n is the number of data, i is the i th data point representing specific locations where GPS is measured and GIA is modeled, w is the weighting factor defined by the inverse of the squared GPS uncertainty (σ), and v_{gps} and v_{gia} are the vertical velocities of the calculated from GPS measurements and modeled GIA estimates.

$$RSS = \sqrt{\sigma_{tg}^2 + \sigma_{sa}^2 + \sigma_{gps}^2}$$

where σ_{tg} is the tide gauge uncertainty, σ_{sa} is the uncertainty associated with satellite altimetry measurements, and σ_{gps} is the GPS error.

$$E = \sqrt{m^2 - R^2}$$

where E is the additive error unaccounted for in the data uncertainties, m is the model misfit, and R is the RSS of the data uncertainties.

$$x(t) = a + bt + c_1 \sin(2\pi t f) + c_2 \cos(2\pi t f) + c_3 \sin(4\pi t f) + c_4 \cos(4\pi t f) + c_5 H(t - t_d) + c_6 H(t - t_v) t$$

Where x is position, t is time, a is the intercept, b is the slope, c_{1-6} are coefficients, f is the frequency, H is the Heaviside function, t_d is the time of displacement, and t_v is the time of a change in velocity.